An Algorithm for Frequency-Domain Noise Analysis in Nonlinear Systems

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ABSTRACT

Many mixed-technology and mixed-signal designs require the assessment of the system's performance in the presence of signals that are best characterized as having continuous frequency spectra: for instance, the simulation of a circuit's behavior in the presence of electrical noise. This paper describes an algorithm for frequency-domain simulation of nonlinear systems capable of handling signals with continuous spectra. The algorithm is based on an orthogonal series expansion of the signals in the frequency domain. Thus it is, in a sense, the dual approach to frequency-domain simulation with respect to harmonic balance, which relies on a time-domain series expansion of the signals. Signal spectra are obtained from the solution of a system of nonlinear algebraic equations whose dimension increases with the desired spectral accuracy. Numerical results obtained on an optical amplifier are presented.

Categories and Subject Descriptors

J.6 [**Computer-Aided Engineering**]: Computer-Aided Design; I.6.8 [**Simulation and Modeling**]: Types of Simulation—*continuous*

General Terms

Algorithms, Verification

Keywords

Circuit simulation, frequency-domain analysis

1. INTRODUCTION

The steady increase in the number of systems based on mixed-technology designs, combined with the continuing trend in the decrease of their physical dimensions, is bringing renewed attention to algorithms and tools capable of assessing circuit or system performance in the presence of noise.

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Most methods developed so far to perform noise analysis in nonlinear systems assume that the perturbation created by the noise is sufficiently small for the system to respond in a linear way to noise interference [1, 2, 3, 4]. Hence they cannot be used if the changes in the system response caused by the interfering signal are not small enough to justify the linearization of the system equations.

The only existing frequency-domain simulation methods that can fully handle nonlinearities rely on the principle of harmonic balance [5], and a number of noise-analysis algorithms based on this approach have been published [6, 7]. In harmonic balance, the solution to the system equations is expressed as a periodic or quasi-periodic Fourier series in the time domain. The drawback intrinsic in this approach is that, by construction, it can only handle signals with discrete frequency spectra. There are many situations, however, in which it is necessary to simulate the behavior of a system in the presence of signals that do not satisfy this requirement.

This paper presents an algorithm suitable for the simulation of nonlinear systems in the frequency domain when the signals involved have continuous spectra. The algorithm is based on a Fourier series expansion of the signals in the frequency domain. Thus it can be considered, in a sense, the dual approach to frequency-domain simulation with respect to harmonic balance. The mathematical formulation of the algorithm is given in Section 2, while Section 3 contains an analysis of the error introduced in the signal spectra computed by the algorithm. Finally, results obtained from numerical simulations are presented in Section 4.

2. MATHEMATICAL FORMULATION

Consider a nonlinear system (e.g. an integrated circuit) that is described by the following system of equations:

$$\frac{d}{dt}\mathbf{f}[\mathbf{x}(t), t] + \mathbf{g}[\mathbf{x}(t), t] = \mathbf{u}(t). \tag{1}$$

In addition to electrical circuits, this equation describes many other types of dynamical systems, with the variables taking on various physical meanings.

In the frequency domain, this equation corresponds to the following equivalent equation [5]:

$$j\omega\mathbf{F}[\mathbf{X}(\omega)] + \mathbf{G}[\mathbf{X}(\omega)] = \mathbf{U}(\omega), \tag{2}$$

where $\mathbf{X}(\omega)$ and $\mathbf{U}(\omega)$ represent the Fourier transforms of $\mathbf{x}(t)$ and $\mathbf{u}(t)$. The traditional approaches to frequency-domain analysis, which are based on harmonic balance, perform sinusoidal expansions of the signals in the time domain. This type of expansion, however, is applicable only to signals that have discrete spectra. This section describes an algorithm developed specifically for frequency-domain simulation of nonlinear systems when the signals have continuous spectra. It is based on a sinusoidal expansion of the signals *in the frequency domain.*

It will be assumed that both the input $\mathbf{u}(t)$ and the solution $\mathbf{x}(t)$ have a frequency spectrum that has a finite upper bound ω_M, as shown in Fig. 1. In practice, this condition can always be satisfied by choosing ω_M sufficiently large. Moreover, it will also be assumed that the spectra of $\mathbf{f}[\mathbf{x}(t), t]$ and $\mathbf{g}[\mathbf{x}(t), t]$ satisfy the same condition.

In many instances, $\mathbf{u}(t)$ and $\mathbf{x}(t)$ contain a DC component, which creates a spike[1] in their spectra at $\omega = 0$. The presence of spikes makes it problematic from a numerical standpoint to expand the spectrum in a sinusoidal series. For this reason, it is more convenient to rewrite (1) so that the DC component is separated from the signals. For this purpose, let $\mathbf{u}_0, \mathbf{x}_0, \mathbf{f}_0, \mathbf{g}_0$ be the DC components of $\mathbf{u}(t), \mathbf{x}(t), \mathbf{f}[\mathbf{x}(t), t]$ and $\mathbf{g}[\mathbf{x}(t), t]$, respectively, and define the following quantities:

$$\begin{array}{ll} \Delta\mathbf{x}(t) = \mathbf{x}(t) - \mathbf{x}_0, & \Delta\mathbf{f}[\Delta\mathbf{x}(t), t] = \mathbf{f}[\mathbf{x}_0 + \Delta\mathbf{x}(t), t] - \mathbf{f}_0, \\ \Delta\mathbf{u}(t) = \mathbf{u}(t) - \mathbf{u}_0, & \Delta\mathbf{g}[\Delta\mathbf{x}(t), t] = \mathbf{g}[\mathbf{x}_0 + \Delta\mathbf{x}(t), t] - \mathbf{g}_0. \end{array}$$

Since $\mathbf{u}_0 = \mathbf{U}(0)$ and $\mathbf{g}_0 = \mathbf{G}[\mathbf{X}(0)]$, setting $\omega = 0$ in (2) shows that $\mathbf{g}_0 = \mathbf{u}_0$. Therefore the following equation is equivalent to (1):

$$\frac{d}{dt}\Delta\mathbf{f}[\Delta\mathbf{x}(t), t] + \Delta\mathbf{g}[\Delta\mathbf{x}(t), t] = \Delta\mathbf{u}(t). \tag{3}$$

Note that this is *not* a linearization of (1) around the DC component of the signals: it is easy to verify that $\mathbf{x}(t) = \mathbf{x}_0 + \Delta\mathbf{x}(t)$ solves (1) exactly, if $\Delta\mathbf{x}(t)$ is the solution of (3).

By assumption, the spectra of all the terms in (3) are contained in the interval $[-\omega_M,\ \omega_M]$. Therefore they can be expanded in an orthogonal series over that interval. For example:

$$\Delta\mathbf{U}(\omega) = \sum_{k=-\infty}^{\infty} \mathbf{U}_{-k} e^{-jk\pi\omega/\omega_M} r(\omega/\omega_M), \tag{4}$$

where the function $r(x)$ is defined as:

$$r(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1. \end{cases}$$

Taking the inverse Fourier transform of (4) yields the following series expansion for $\Delta\mathbf{u}(t)$:

$$\Delta\mathbf{u}(t) = \frac{\omega_M}{\pi} \sum_{k=-\infty}^{\infty} \mathbf{U}_{-k}\, \mathrm{sync}[\omega_M(t - t_k)], \tag{5}$$

where: $\mathrm{sync}\, x = \sin x / x$, and $t_k = k\pi/\omega_M$. Evaluating this expression at $t = t_n$ shows that coefficients $\mathbf{U}_k$ in (4) are related to the values of $\Delta\mathbf{u}(t)$ by the following relationship:

$$\Delta\mathbf{u}(t_n) = \frac{\omega_M}{\pi}\mathbf{U}_{-n}. \tag{6}$$

Equations (5) and (6), taken together, express the well-known sampling theorem [8].

[1]Theoretically, a Dirac impulse.

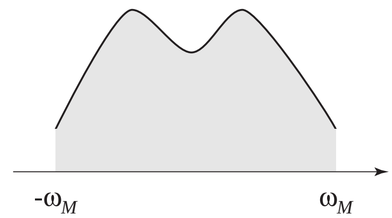

Figure 1: Continuous frequency spectrum

A similar series expansion can be obtained for the left-hand side of (3):

$$\begin{aligned} &\frac{d}{dt}\Delta\mathbf{f}[\Delta\mathbf{x}(t), t] + \Delta\mathbf{g}[\Delta\mathbf{x}(t), t] \\ &= \sum_{k=-\infty}^{\infty} (\Delta\dot{\mathbf{f}}[\Delta\mathbf{x}(t_k), t_k] + \Delta\mathbf{g}[\Delta\mathbf{x}(t_k), t_k])\, \mathrm{sync}[\omega_M(t - t_k)], \end{aligned}$$

where: $\Delta\dot{\mathbf{f}}[\Delta\mathbf{x}(t_k), t_k] = \left.\frac{d}{dt}\Delta\mathbf{f}[\Delta\mathbf{x}(t), t]\right|_{t=t_k}$. Consequently, in order for (3) to be satisfied the following equality must be satisfied at all points t_k:

$$\Delta\dot{\mathbf{f}}[\Delta\mathbf{x}(t_k), t_k] + \Delta\mathbf{g}[\Delta\mathbf{x}(t_k), t_k] = \Delta\mathbf{u}(t_k). \tag{7}$$

An expression for $\Delta\dot{\mathbf{f}}[\mathbf{x}(t_k), t_k]$ can be obtained starting from the following series expansion:

$$\Delta\mathbf{f}[\Delta\mathbf{x}(t), t] = \sum_{i=-\infty}^{\infty} \Delta\mathbf{f}[\Delta\mathbf{x}(t_i), t_i]\, \mathrm{sync}[\omega_M(t - t_i)].$$

Differentiating both sides of this equation yields:

$$\Delta\dot{\mathbf{f}}[\Delta\mathbf{x}(t_k), t_k] = \omega_M \sum_{i=-\infty}^{\infty} \Delta\mathbf{f}[\Delta\mathbf{x}(t_i), t_i]\, \mathrm{sync}'[\omega_M(t_k - t_i)].$$

But:

$$\begin{aligned} \mathrm{sync}'[\omega_M(t_k - t_i)] &= \mathrm{sync}'[\pi(k - i)] \\ &= \begin{cases} 0, & i = k \\ \dfrac{(-1)^{k-i}}{\pi(k-i)}, & i \ne k. \end{cases} \end{aligned}$$

Therefore:

$$\Delta\dot{\mathbf{f}}[\Delta\mathbf{x}(t_k), t_k] = \frac{\omega_M}{\pi} \sum_{\substack{i=-\infty \\ i \ne k}}^{\infty} \Delta\mathbf{f}[\Delta\mathbf{x}(t_i), t_i] \frac{(-1)^{k-i}}{k - i}, \tag{8}$$

where the value $i = k$ is excluded from the summation. Substituting this expression for $\Delta\dot{\mathbf{f}}[\Delta\mathbf{x}(t_k), t_k]$ in (7), and truncating the infinite series to a finite number of terms (which is dependent upon the accuracy that one wishes to achieve), yields the following equation:

$$\begin{aligned} \frac{\omega_M}{\pi} \sum_{\substack{i=-N \\ i \ne k}}^{N} \Delta\mathbf{f}[\Delta\mathbf{x}(t_i), t_i] \frac{(-1)^{k-i}}{k - i} \\ + \Delta\mathbf{g}[\Delta\mathbf{x}(t_k), t_k] &= \Delta\mathbf{u}(t_k). \end{aligned} \tag{9}$$

This is a set of $2N + 1$ algebraic equations in the $2N + 1$ unknowns $\Delta\mathbf{x}(t_{-N}), \Delta\mathbf{x}(t_{-N+1}), \ldots, \Delta\mathbf{x}(t_N)$, which can be solved using Newton's method. The spectrum of $\Delta\mathbf{x}(t)$ can then be computed from relationships analogous to those in (4) and (6):

$$\Delta \mathbf{X}_N(\omega) = \frac{\pi}{\omega_M} \sum_{k=-N}^{N} \Delta \mathbf{x}(t_k) e^{-jk\pi\omega/\omega_M} r(\omega/\omega_M). \quad (10)$$

3. ERROR ANALYSIS

The algorithm described in the previous section replaces $\Delta\dot{\mathbf{f}}[\Delta\mathbf{x}(t_k), t_k]$ with a linear combination of the values of $\Delta\mathbf{f}[\Delta\mathbf{x}(t), t]$ at other timepoints, as shown in (8). This is the same type of approximation that is made in linear multistep numerical integration methods. This suggests that an in-depth analysis of the error introduced by the algorithm when the system is linear can already provide useful insight into the algorithm's numerical performance.

If the system is linear, equation (1) becomes:

$$\mathbf{F}\frac{d\mathbf{x}}{dt} + \mathbf{G}\mathbf{x}(t) = \mathbf{u}(t), \quad (11)$$

or, equivalently, in the frequency domain:

$$(j\omega\mathbf{F} + \mathbf{G})\mathbf{X}(\omega) = \mathbf{U}(\omega). \quad (12)$$

The exact solution of this equation is:

$$\mathbf{X}(\omega) = (j\omega\mathbf{F} + \mathbf{G})^{-1}\mathbf{U}(\omega). \quad (13)$$

Let $F(\omega)$ be a generic function vanishing outside the interval $[-\omega_M,\ \omega_M]$ of Fig. 1. Let $P_N F$ denote the function obtained by truncating the Fourier expansion of F to the N-th term:

$$P_N F(\omega) = \sum_{k=-N}^{N} F_k e^{jk\pi\omega/\omega_M} r(\omega/\omega_M),$$

and let: $R_N F(\omega) = F(\omega) - P_N F(\omega)$. The following result can be established.

THEOREM 1. *The spectrum $\mathbf{X}_N(\omega)$ obtained by applying (9) and (10) to (11) satisfies the following equation:*

$$P_N[(j\omega\mathbf{F} + \mathbf{G})\mathbf{X}_N(\omega)] = P_N\mathbf{U}(\omega). \quad (14)$$

Proof: Because of space limitations, the proof of this theorem is omitted. ■

An equation for the error: $\mathbf{E}_N(\omega) = \mathbf{X}(\omega) - \mathbf{X}_N(\omega)$ can now be obtained by subtracting (14) from (12):

$$\begin{aligned} R_N\mathbf{U} &= \mathbf{U} - P_N\mathbf{U} \\ &= (j\omega\mathbf{F} + \mathbf{G})\mathbf{X} - P_N[(j\omega\mathbf{F} + \mathbf{G})\mathbf{X}_N] \\ &= (j\omega\mathbf{F} + \mathbf{G})(\mathbf{X} - \mathbf{X}_N) + (j\omega\mathbf{F} + \mathbf{G})\mathbf{X}_N \\ &\quad - P_N[(j\omega\mathbf{F} + \mathbf{G})\mathbf{X}_N] \\ &= (j\omega\mathbf{F} + \mathbf{G})\mathbf{E}_N + R_N[(j\omega\mathbf{F} + \mathbf{G})\mathbf{X}_N]. \end{aligned}$$

Consequently:

$$\begin{aligned} &(j\omega\mathbf{F} + \mathbf{G})\mathbf{E}_N(\omega) \\ &\quad = R_N\mathbf{U}(\omega) - R_N[(j\omega\mathbf{F} + \mathbf{G})\mathbf{X}_N(\omega)]. \quad (15) \end{aligned}$$

Equation (15) is similar to (12), but its right-hand side consists of the sum of two terms, both of which represent errors introduced by truncating an infinite Fourier series to a finite number of terms. The first term derives from the truncation of the Fourier series expansion of $\mathbf{U}(\omega)$, and the second from that of $(j\omega\mathbf{F} + \mathbf{G})\mathbf{X}_N(\omega)$. In particular, $\mathbf{E}_N$ may be non-zero even if $R_N\mathbf{U} = 0$, i.e. even if the right-hand side of (12) is expressed exactly by a finite Fourier sum.

Further insight into the error introduced by truncating an infinite Fourier series can be obtained by deriving an explicit expression for $P_N F(\omega)$ in the following way. From the well-known formula for the Fourier coefficients:

$$F_k = \frac{1}{2\omega_M} \int_{-\omega_M}^{\omega_M} F(w) e^{-jk\pi w/\omega_M}\, dw$$

it follows that:

$$\begin{aligned} P_N F(\omega) &= \sum_{k=-N}^{N} F_k e^{jk\pi\omega/\omega_M} = \\ &\frac{1}{2\omega_M} \int_{-\omega_M}^{\omega_M} F(w) \left(\sum_{k=-N}^{N} e^{jk\pi(\omega - w)/\omega_M} \right) dw. \end{aligned}$$

But:

$$\sum_{k=-N}^{N} e^{jkx} = \frac{\sin\left(N + \frac{1}{2}\right)x}{\sin(x/2)}.$$

Therefore:

$$\begin{aligned} &P_N F(\omega) = \\ &\quad \frac{1}{2\omega_M} \int_{-\omega_M}^{\omega_M} F(w) \frac{\sin\left(N + \frac{1}{2}\right)\pi(\omega - w)/\omega_M}{\sin[\pi(\omega - w)/2\omega_M]}\, dw. \end{aligned}$$

This expression for $P_N F(\omega)$ shows that truncating an infinite Fourier series to a finite number of terms is equivalent to performing a convolution of the original function $F(\omega)$ with the following function:

$$W_N(\omega) = \frac{1}{2\omega_M} \frac{\sin\left(N + \frac{1}{2}\right)\pi\omega/\omega_M}{\sin(\pi\omega/2\omega_M)}.$$

The oscillatory nature of $W_N(\omega)$ can potentially create spurious oscillations in the computed spectrum (Gibbs phenomenon). This effect, known as *windowing* in the spectral analysis of signals, occurs whenever the Fourier transform of a signal that is theoretically infinitely long is computed numerically using a finite length FFT. In particular, the function $W_N(\omega)$ obtained above corresponds to the application of a rectangular window to the time-domain signal. This observation suggests that the numerical performance of the algorithm described in Section 2 can be improved using windowing techniques developed for the spectral analysis of signals [9].

4. NUMERICAL RESULTS

The numerical performance of the algorithm described in the previous sections was verified by performing simulations of a number of test circuits. The results obtained on a CMOS optical amplifier, shown in Fig. 2, are presented next.

This circuit amplifies the output of a photodetector, modeled by a current source applied to its input. The amplifier load is modeled by a 30 kΩ resistor at the amplifier's output. A signal having a uniform power spectral density within a 1 GHz bandwidth (white noise) was applied to the amplifier's input. The spectrum of the resulting output signal, computed by the algorithm with $N = 128$, is shown in Fig. 3. The computation required approximately 32 minutes of CPU time on a Sun Ultra 60 workstation running Solaris 8. The ripple that is clearly visible in the graph is a manifestation of the Gibbs phenomenon mentioned earlier,

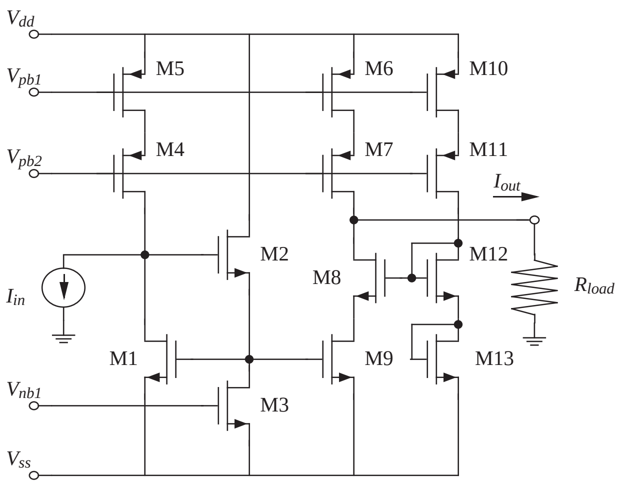

Figure 2: A CMOS optical amplifier

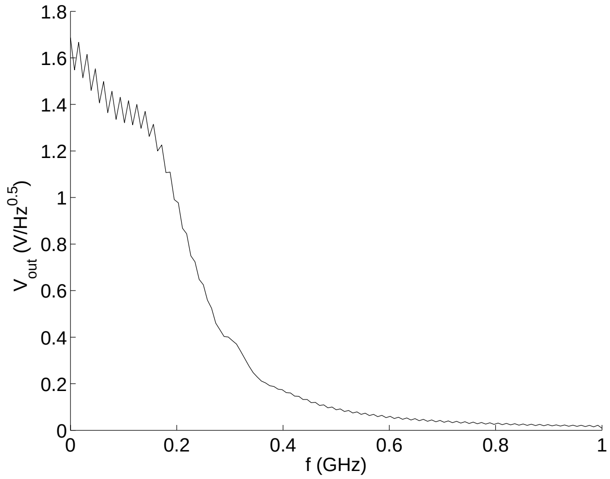

Figure 3: Output spectrum of optical amplifier

and it is caused by the abrupt truncation of an infinitely long signal to one of finite length. It can be reduced or eliminated by using appropriate windowing or curve fitting techniques.

5. CONCLUSION

The approach to frequency-domain simulation described in this paper is, in a way, the dual of the well-known analysis method based on harmonic balance, in that the unknown signals are expanded in a sinusoidal series in the frequency domain, instead of the time domain. Taking the inverse Fourier transform of the series expansion transforms the differential equation describing the system into a finite set of algebraic equations in the time domain (instead of the frequency domain, as with harmonic balance). The number of equations is chosen based on the desired accuracy, just as in harmonic balance the number of harmonics to be solved for is determined by the desired accuracy. Thus the two methods are one the mirror image of the other, but, unlike harmonic balance, the algorithm described in this paper is suitable for handling signals with continuous spectra. As with harmonic balance, the computational effort required for the analysis increases with the desired accuracy: in this case, the resolution in the computed spectrum. Specifically, the size of the system generated by (9) is $(2N+1)D$, where D is the dimension of $\mathbf{x}(t)$. Thus the size of this system increases linearly with the number of terms in (10), and can become very large even for circuits of relatively modest size. This problem can be alleviated by the use of iterative solution methods that do not require matrix inversion [10].

6. ACKNOWLEDGEMENTS

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