

# An Introduction to Statistics for EC Experimental Analysis



Authors: Mark Wineberg  
Steffen Christensen



## Top 5 Experimental Analysis Myths in EC

- i. Results from 1 run is all that is needed
  - No, shows only proof of concept
- ii. The best value achieved in a set of runs tells you something about the population distribution
  - No
- iii. Using the same random number generator seed for both systems provides a fairer comparison
  - It doesn't - it's the statistical properties of the system that we are looking for
- iv. One system is obviously better than the other when looking at the data or graph - no statistics necessary
  - If it is so obvious, then will be easy to show statistically
  - might as well do the stats
  - shows that you are objectively confident in your conclusion
- v. "My average is better than yours" means "my technique is better than yours"
  - In the best case you would need to take variance into account



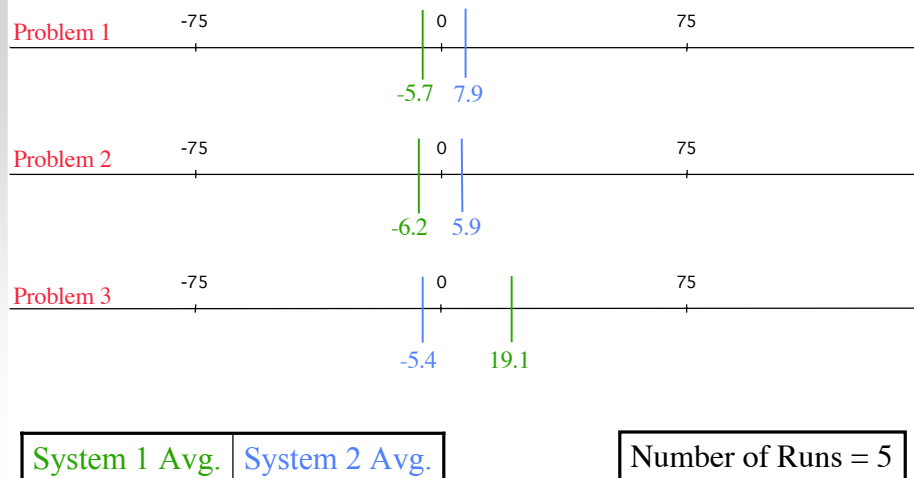
## Myth: Averages are Everything

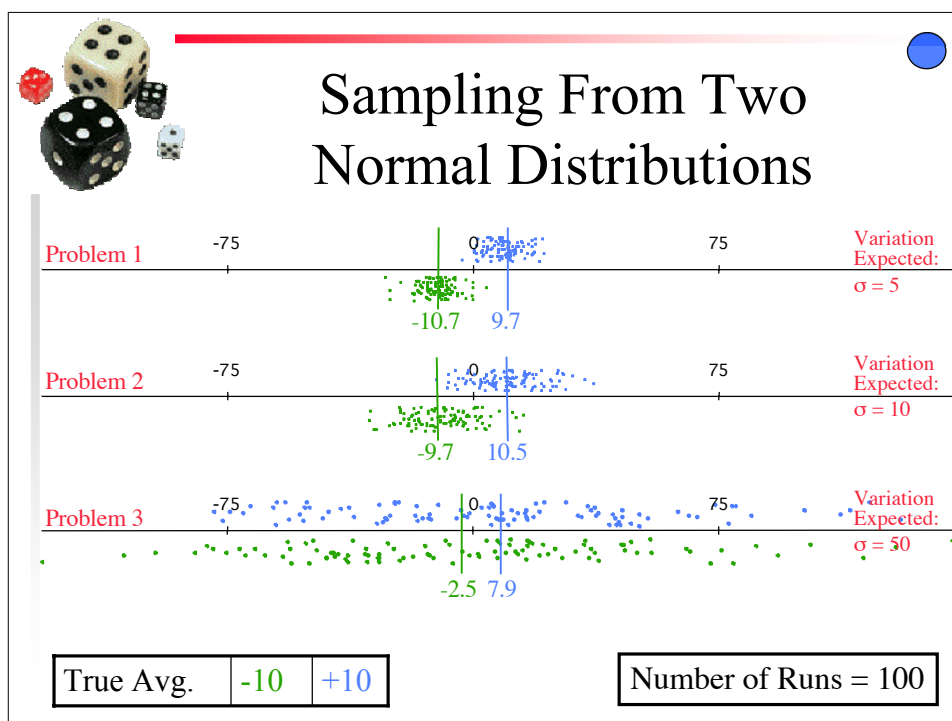
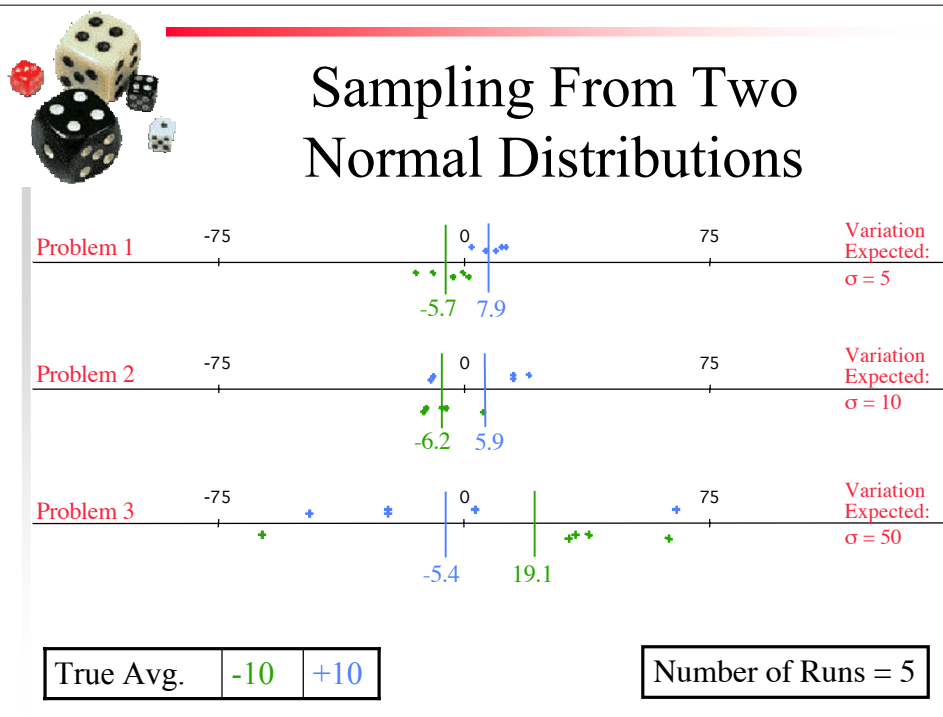
- We might get unlucky with our data distribution – a simple comparison between two averages might not give the same result as the comparison between two distributions
- We will demonstrate with a few examples
- Consider the following samples of two distributions (blue and green), which are normally distributed and are drawn from the distributions with these parameters:

| Mean | StdDev               | Reps       |
|------|----------------------|------------|
| +10  | $\sigma = 5, 10, 50$ | N = 100, 5 |
| -10  | $\sigma = 5, 10, 50$ | N = 100, 5 |



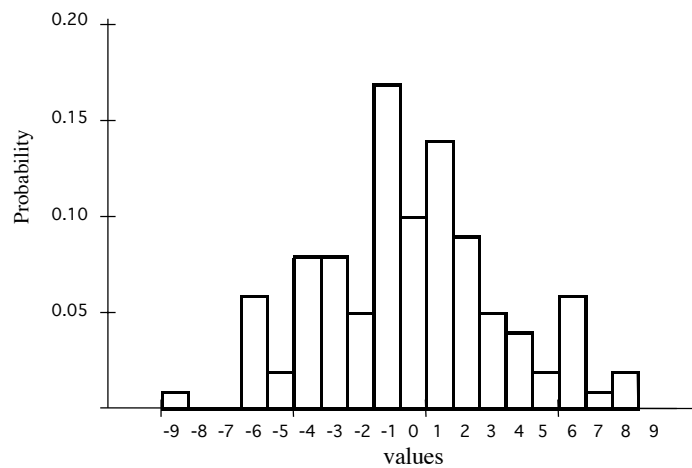
## Sampling From Two Normal Distributions





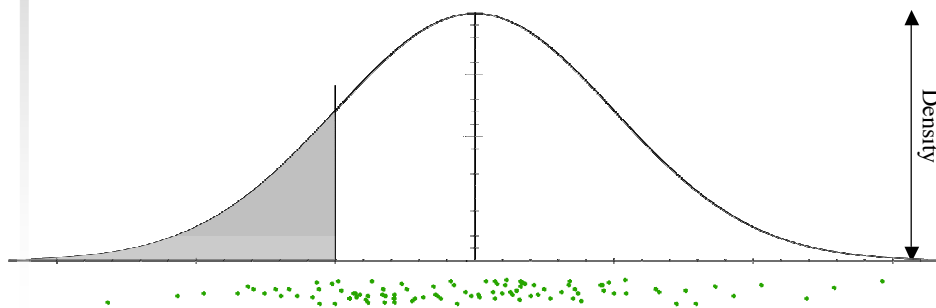


## Probability Function



## Probability Density Function

- The greater the density of observed outcomes, the larger the density function's value
- Important: \*\*\* Probability = integration (sum) of the density function within an interval \*\*\*





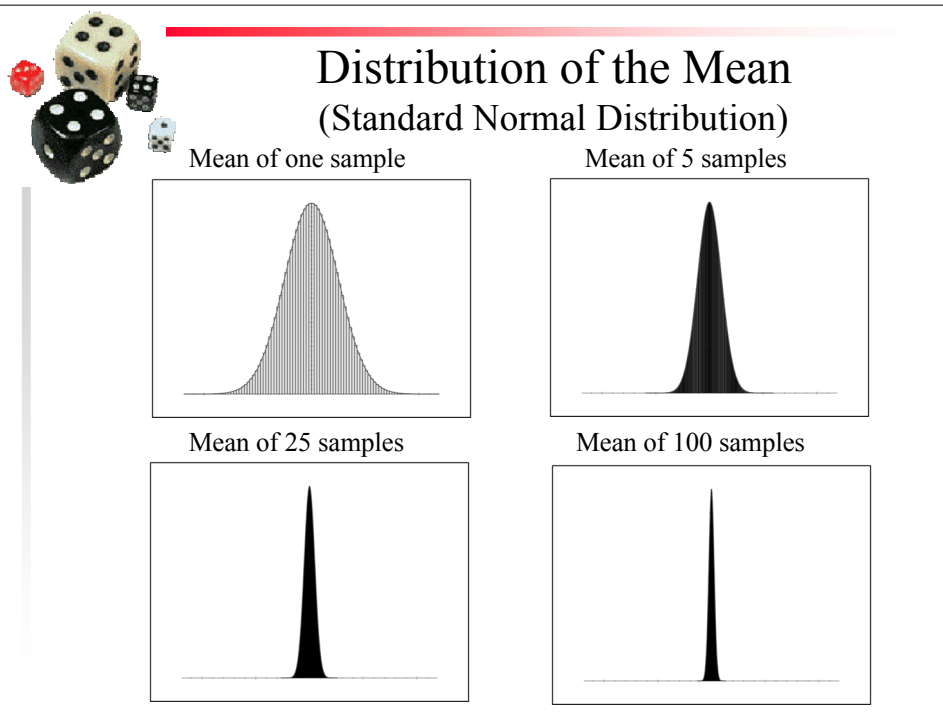
## What Are We Interested In?

- For most statistical analysis for EC the question is
  - Is my new way better than the old way?
  - Statistically this translates into a statement about the difference between means: “Is the difference between ‘my mean’ and ‘the old mean’ greater than zero?”
- We will approach this question in 2 steps:
  1. What can we say about the true mean of a *single* distribution?
  2. How can we compare the true means of *two* or more distributions?



## Distribution of the Mean

- Consider the distribution of the average of a set of  $n$  independent samples
  - If  $n = 1$ , the distribution of the average is just the distribution itself, since we have only the single data point
  - If  $n$  is larger than one, the distribution of the mean must be narrower than the distribution of the population
    - i.e. the variance and standard deviation must be smaller
  - In fact, the standard deviation of the mean of  $n$  samples is given by  $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$



## Confidence Intervals

- As the “finger” gets narrower, the mean of the samples approaches the true mean
- We’d like to say that in the overwhelming majority of all possible experiments, the true mean of this distribution will lie within a specified interval
  - Example: In 99% of cases, the true mean of the distribution, estimated from our 50 samples, lies within the interval  $[64, 79]$  – called a *confidence interval* for the mean



## $t$ Distribution

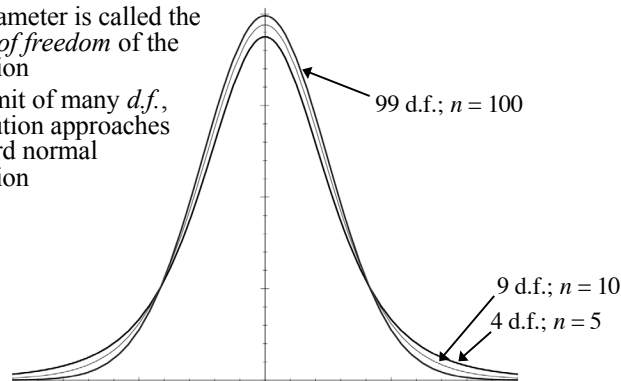
- Of course, we don't know the true mean,  $\mu$ , or true standard deviation,  $\sigma$
- We *do* know the mean of the samples,  $\bar{X}$ , the sample size,  $n$ , and the sample standard deviation,  $s_X$
- If the source distribution is normally distributed, the shape and size of the "finger" is known exactly!
  - We can determine the odds that the true mean lies within a specified range of  $\bar{X}$
  - The distribution of the sample average follows a  $t$  distribution with  $n - 1$  degrees of freedom, where

$$T = \frac{(\bar{X} - \mu)}{s_{\bar{X}}} = \frac{(\bar{X} - \mu)}{s_X / \sqrt{n}}$$



## $t$ Distribution

- The  $t$  "distribution" is really a family of distributions – the shape of the distribution changes as the number of samples,  $n$ , changes
  - This parameter is called the *degrees of freedom* of the distribution
  - In the limit of many *d.f.*,  $t$  distribution approaches a standard normal distribution



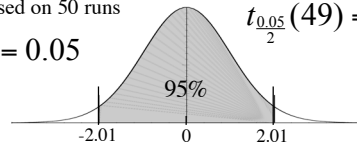


## Estimating the Mean: Confidence Intervals Around the Average

If samples taken from a *standard normal distribution* ( $\mu = 0, \sigma = 1$ ), the sample average has a *t* distribution.

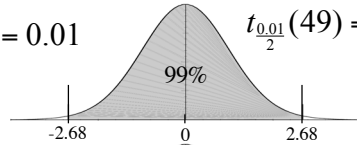
Based on 50 runs

$\alpha = 0.05$



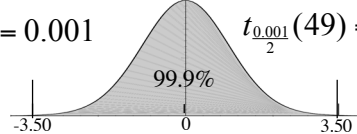
$$t_{\frac{0.05}{2}}(49) = 2.01$$

$\alpha = 0.01$



$$t_{\frac{0.01}{2}}(49) = 2.68$$

$\alpha = 0.001$



$$t_{\frac{0.001}{2}}(49) = 3.50$$

- For Confidence Intervals, we can use cutoff *t* values
- The wider the cutoff values, the more likely the true mean will fall between them
- $\alpha$  is the probability of obtaining values outside the cutoffs
  - Confidence Level is  $1 - \alpha$
- Cut off *t* values can be computed using Excel: **=TINV( $\alpha, n - 1$ )**
  - Note: TINV() is already 2 sided



## Estimating the Mean: Confidence Intervals Around the Average

- We know that

$$T = \frac{(\bar{X} - \mu_X)}{s_X / \sqrt{n}}$$

- Using the  $\pm t_{\frac{\alpha}{2}}(n-1)$  cutoff *t*-values we can form a Confidence Interval that has a  $1 - \alpha$  C.L with  $n - 1$  degrees of freedom
- Substituting the cutoff values from the C.I. into the above equation produces

$$\pm t_{\frac{\alpha}{2}}(n-1) = \frac{(\bar{X} - \mu_X)}{s_X / \sqrt{n}}$$

which can be rewritten as

$$\mu_X = \bar{X} \pm t_{\frac{\alpha}{2}}(n-1) \frac{s_X}{\sqrt{n}}$$



## Estimating the Mean: Confidence Intervals Around the Average

- Confidence Intervals can be written in 3 equivalent ways

### Error Bounds

$$\mu_X = \bar{X} \pm t_{\frac{\alpha}{2}}(n-1) \frac{S_X}{\sqrt{n}}$$

### Confidence Intervals

$$\bar{X} - t_{\frac{\alpha}{2}}(n-1) \frac{S_X}{\sqrt{n}} \leq \mu_X \leq \bar{X} + t_{\frac{\alpha}{2}}(n-1) \frac{S_X}{\sqrt{n}}$$

$$\mu_X \in \left[ \bar{X} - t_{\frac{\alpha}{2}}(n-1) \frac{S_X}{\sqrt{n}}, \bar{X} + t_{\frac{\alpha}{2}}(n-1) \frac{S_X}{\sqrt{n}} \right]$$



## Estimating the Mean: Confidence Intervals Around the Average

Example:

- An EC experimenter runs a GA on a TSP
- At the end of each run, the smallest length tour that had been found during the run was recorded
- The GA is run 50 times on the same TSP problem
- On average the GA found solutions with a tour length of 272
- The standard deviation of these tours is 87
- We want to compute a Confidence Interval using a 99% Confidence level



## Estimating the Mean: Confidence Intervals Around the Average

- From the problem we know that the average GA run produced tours of

$$\bar{X} = 272 \text{ that had } s_X = 87$$

We know that  $\mu_X = \bar{X} \pm t_{\frac{\alpha}{2}}(n-1) \frac{s_X}{\sqrt{n}}$

- Also from the problem  $n = 50$  and  $\alpha = (1 - 0.99) = 0.01$

so the  $\pm t$  cutoff value is  $t_{\frac{0.01}{2}}(50-1) = t_{\frac{0.01}{2}}(49)$

using Excel we see that TINV(0.01,49) is 2.68

so  $\bar{X} = \mu_X \pm 2.68 \frac{s_X}{\sqrt{50}} = \mu_X \pm 0.38s_X$

and so  $239 \leq \mu_X \leq 305$  with a 99% C.L.

i.e. there is only a 1% chance that the true mean lies outside the confidence interval formed around average



## Using Confidence Intervals to Determine Whether My Way is Better

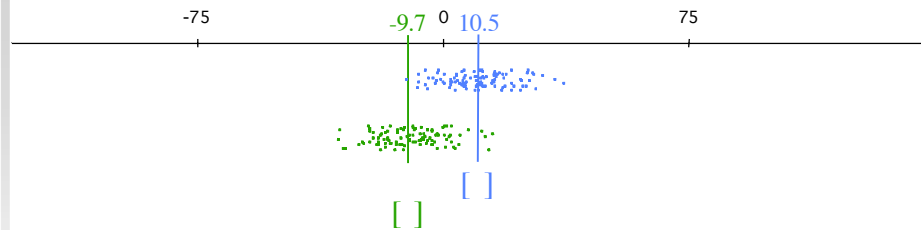
If we have two different EC systems how can we tell if one is better than the other?

Trivial method: Find confidence intervals around both means

- If the CIs don't overlap
  - Then it is a rare occurrence when the two systems do have identical means
  - The system with the better mean can be said to be better on average with a probability better than the Confidence Level
- If the CIs do overlap
  - Can't say that the two systems are different with this technique
  - Either:
    - The two systems are equivalent
    - We haven't sampled enough to discriminate between the two



## Confidence Interval Example

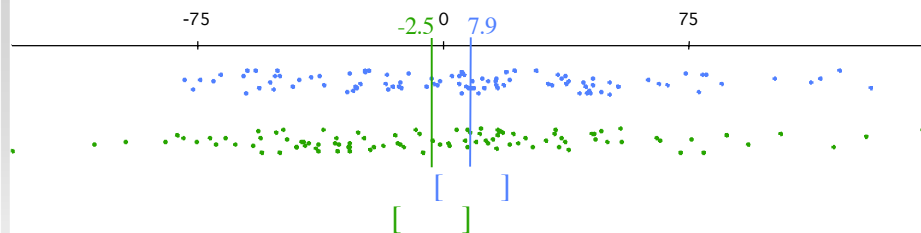


| $\mu$ | $\sigma$ |
|-------|----------|
| +10   | 10       |
| -10   | 10       |

| 95% Confidence Level |           |               |                                     |       |       |
|----------------------|-----------|---------------|-------------------------------------|-------|-------|
| $n$                  | $\bar{X}$ | $s_{\bar{X}}$ | $1.96 \frac{s_{\bar{X}}}{\sqrt{n}}$ | Lower | Upper |
| 100                  | 10.5      | 10.0          | 3.3                                 | 7.2   | 13.8  |
| 100                  | -9.7      | 10.1          | 3.3                                 | -13.1 | -6.4  |



## Confidence Interval Example



| $\mu$ | $\sigma$ |
|-------|----------|
| +10   | 50       |
| -10   | 50       |

| 95% Confidence Level |           |               |                                     |       |       |
|----------------------|-----------|---------------|-------------------------------------|-------|-------|
| $n$                  | $\bar{X}$ | $s_{\bar{X}}$ | $1.96 \frac{s_{\bar{X}}}{\sqrt{n}}$ | Lower | Upper |
| 100                  | 7.9       | 47.1          | 9.2                                 | -1.3  | 17.1  |
| 100                  | -2.5      | 52.1          | 10.2                                | -12.7 | 7.7   |



## Assumptions, assumptions

- All we have said so far applies only if the source distribution is a normal distribution
- What if the source distribution is not a normal distribution?
  - In EC, the source distribution is essentially *never* normal!
- Fortunately, there is one nice property that can help us out
  - The *Central Limit Theorem*: the sum of many identically distributed random variables tends to a Gaussian
  - Equation of the mean:

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n x_i$$

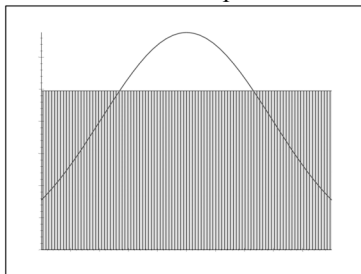
- So the mean of any set of samples tends to a normal distribution



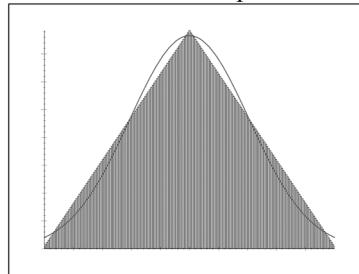
## Central Limit Theorem

- The sum of many *independent, identically distributed (IID)* random variables approaches a Gaussian normal curve
- E.g. Uniform distribution on  $[0, 1]$ :

Mean of one sample



Mean of two samples

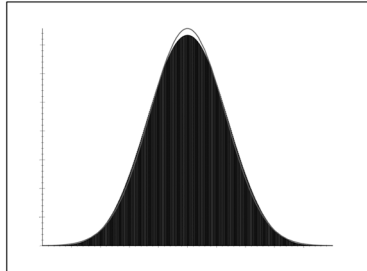




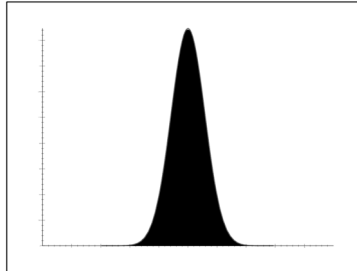
## Central Limit Theorem

- E.g. Uniform distribution (continued):

Mean of five samples

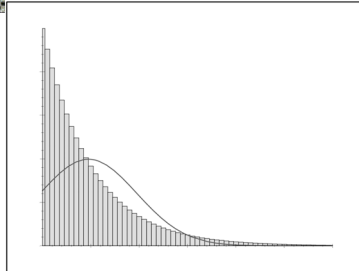


Mean of 25 samples

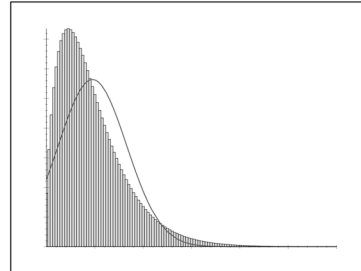


## Exponential Distribution

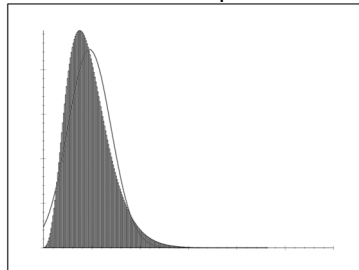
Mean of one sample



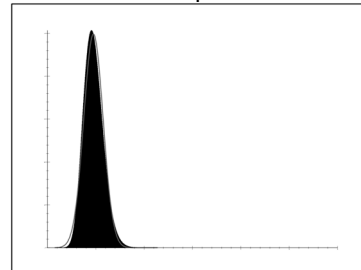
Mean of two samples



Mean of five samples



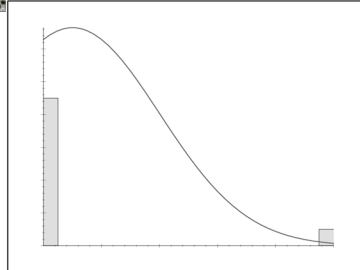
Mean of 25 samples



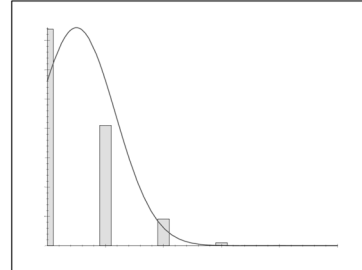


## Binomial Distribution ( $p = 0.1$ )

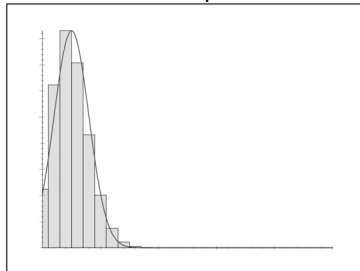
Mean of one sample



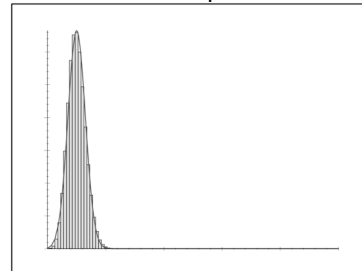
Mean of five samples



Mean of 25 samples



Mean of 100 samples



## Improving the Sensitivity: The Student $t$ Test

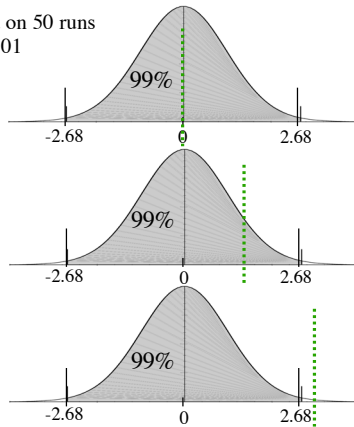
- The Student  $t$  Test is the basic test used in statistics
  - Idea: Gain sensitivity by looking at the difference between the means of the two systems
  - If there is no difference between the actual means of the 2 systems
    - then the difference between the sample averages should be 0, with some error that should follow the  $t$  distribution
    - this is because the difference btw 2 normal distributions is also normal
    - so the sample average should be a  $t$  distribution as usual
  - now we can see if the computed difference of the sample averages falls outside a confidence interval (for some  $\alpha$ ) for the  $t$  distribution



## The Student $t$ Test

Where the normalized difference falls on the  $t$  distribution determines whether difference expected if both systems were actually performing the same

Based on 50 runs  
 $\alpha = 0.01$



- Normalized difference called the  $t$  score

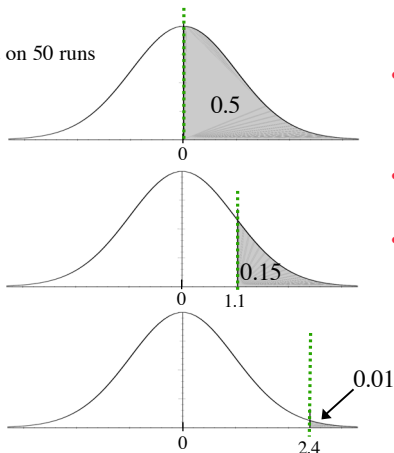
$$t \text{ score} = \frac{\bar{X}_2 - \bar{X}_1}{\sqrt{\frac{s_{X_1}^2}{n_1} + \frac{s_{X_2}^2}{n_2}}}$$

- Distribution again differs for different sample sizes
  - Degrees of Freedom is now  $= n_1 + n_2 - 2$
- $t$  test either succeeds or fails
  - $t$  score greater than cutoff for a given C.L. or not



## The Student $t$ Test: $p$ -values

Based on 50 runs



- The cut-off values produces a binary decision: true or false
  - loses information
- Better to report the probability that two systems are different
- This is the complement of the probability that they are the same
  - $1 - \Pr(T < t \text{ score})$
  - Called the  $p$ -value



## $t$ Test Step by Step

1. Compute the 2 averages  $\bar{X}_1$  and  $\bar{X}_2$
2. Compute standard deviations  $s_1$  and  $s_2$
3. Compute degrees of freedom:  $n_1 + n_2 - 2$
4. Calculate  $T$  statistic:  $T = \frac{(\bar{X}_1 - \bar{X}_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$
5. Compute the  $p$ -value
  - $p$ -value = the area under the  $t$  distribution outside  $[-T, T]$
  - Use **=TDIST( $T, n_1 + n_2 - 2, 2$ )** in Excel
    - The final “2” in Excel means “two-sided”



## When The CLT Fails You

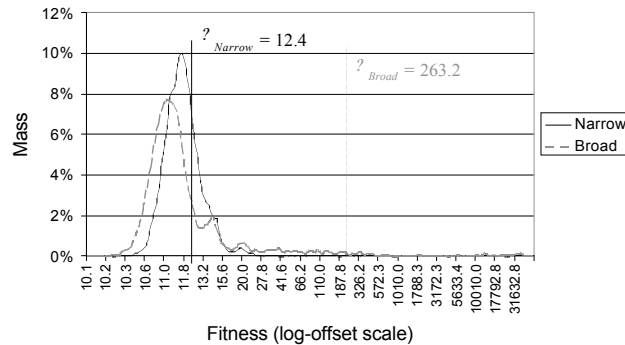
- Everything we have done so far depends on the Central Limit Theorem holding
  - But this is not always true
  - ***In fact for EC it rarely holds***
- Problems occur when
  - ...you have a non-zero probability of obtaining infinity
    - Mean and standard deviation are infinite!
  - ...the sample average depends highly on a few scores
    - When the mean of your distribution is not measuring what you want, consider using the median instead (rank-based statistics)
- EC alert!
  - Many data in evolutionary computing are often highly skewed because some local optima in the search space are very unfit
  - Example follows



## When The CLT Fails You

From a node layout problem where fitness is absolute error (minimization)

- Here are the PDFs of 2 EC parameter settings, named Broad and Narrow for convenience



- Here Broad's mean is much *worse* than Narrow's because of its extended tail, even though Broad often beats Narrow in practice!
  - We don't really care about the 8% of trials where Broad performs badly



## So what should we do?

- There are tests that use Ranks instead of actual values
  - These are called **Non-Parametric Tests**
  - They measure how interspersed the samples from the two treatments are
    - If the result is "alternating" it is assumed that there is no effective difference



## Non-Parametric Tests

- Ranks are uniformly distributed  
(think of percentiles – uniform on  $[0\%, 100\%] = [0, 1]$ )
- The sum of ranks and average of ranks will be approximately normally distributed because of the Central Limit Theorem, as long as we have 5 or more samples
  - This result is independent of the particular distributions of the 2 treatments
  - So we can perform a  $t$  test on the ranks
- 2 other techniques with similar results are commonly seen
  - Wilcoxon's Rank-Sum test
  - Mann-Whitney U test
- All are effectively equivalent, and the test is often called the "Mann-Whitney-Wilcoxon test" by statisticians



## How To Rank the Data

- Augment each data point with a treatment identifier and an additional slot for its rank
- Sort the data sets together by value
  - record the ranks of all values in their rank slot
  - assign the average rank of tied values to each tied value
- Resort by the original order thus splitting the data sets back out
  - keep the combined ranking with each data point
- Apply your  $t$  test on the ranked values



|   |      |
|---|------|
| A | 0.03 |
| A | 0.91 |
| A | 0.64 |
| A | 0.99 |
| A | 0.64 |
| A | 0.16 |
| A | 0.16 |
| A | 0.91 |
| A | 0.16 |
| A | 0.27 |

Two sets of Data

|   |      |
|---|------|
| B | 0.64 |
| B | 0.08 |
| B | 0.16 |
| B | 0.27 |
| B | 0.02 |
| B | 0.01 |
| B | 0.16 |
| B | 0.03 |
| B | 0.03 |
| B | 0.64 |

## Ranked Example



## Ranked Example

Combine the data into a single array

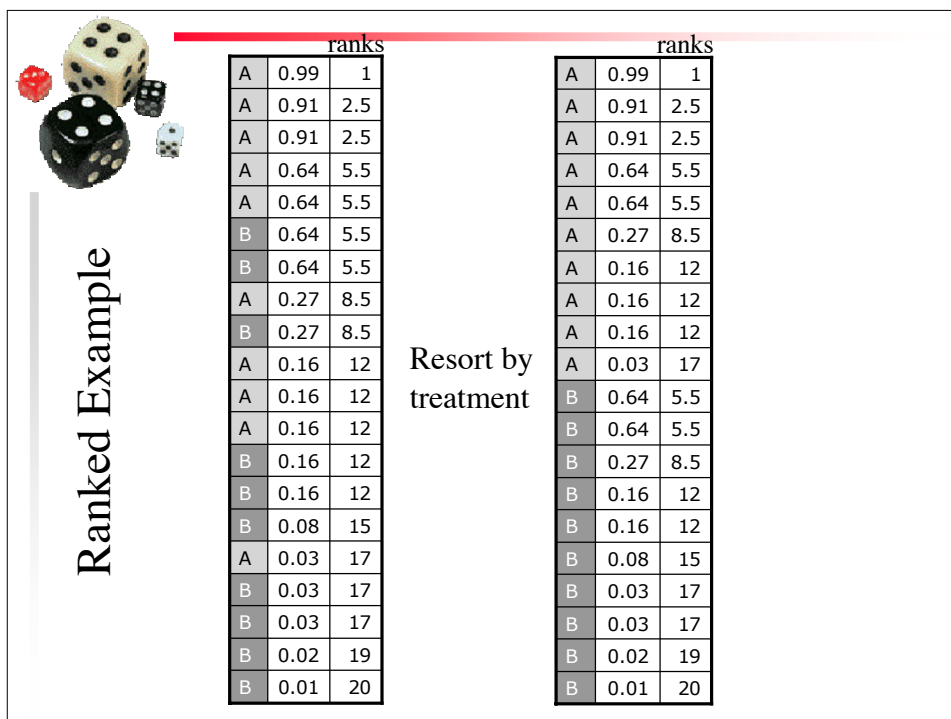
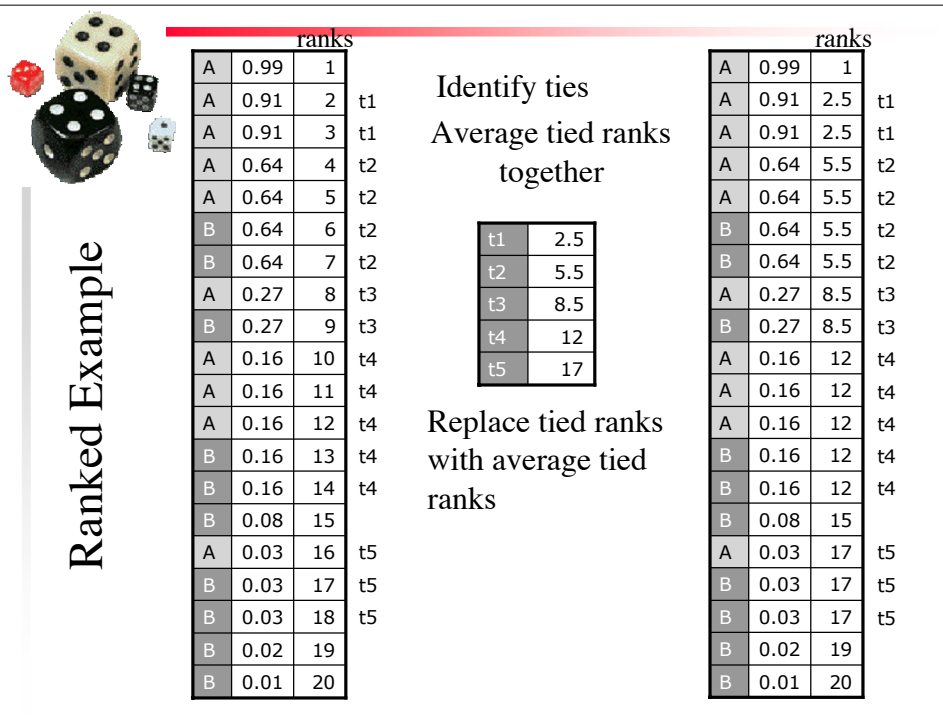
|   |      |
|---|------|
| A | 0.03 |
| A | 0.91 |
| A | 0.64 |
| A | 0.99 |
| A | 0.64 |
| A | 0.16 |
| A | 0.16 |
| A | 0.91 |
| A | 0.16 |
| A | 0.27 |
| B | 0.64 |
| B | 0.08 |
| B | 0.16 |
| B | 0.27 |
| B | 0.02 |
| B | 0.01 |
| B | 0.16 |
| B | 0.03 |
| B | 0.03 |
| B | 0.03 |
| B | 0.64 |

Sort the combined data

|   |      |
|---|------|
| A | 0.99 |
| A | 0.91 |
| A | 0.91 |
| A | 0.64 |
| A | 0.64 |
| B | 0.64 |
| B | 0.64 |
| A | 0.27 |
| B | 0.27 |
| A | 0.16 |
| A | 0.16 |
| A | 0.16 |
| B | 0.16 |
| B | 0.16 |
| B | 0.16 |
| B | 0.08 |
| A | 0.03 |
| B | 0.03 |
| B | 0.03 |
| B | 0.03 |
| B | 0.02 |
| B | 0.01 |

Give each data element its corresponding rank

|   |      |    |
|---|------|----|
| A | 0.99 | 1  |
| A | 0.91 | 2  |
| A | 0.91 | 3  |
| A | 0.64 | 4  |
| A | 0.64 | 5  |
| B | 0.64 | 6  |
| B | 0.64 | 7  |
| A | 0.27 | 8  |
| B | 0.27 | 9  |
| A | 0.16 | 10 |
| A | 0.16 | 11 |
| A | 0.16 | 12 |
| B | 0.16 | 13 |
| B | 0.16 | 14 |
| B | 0.16 | 15 |
| A | 0.03 | 16 |
| B | 0.03 | 17 |
| B | 0.03 | 18 |
| B | 0.03 | 19 |
| B | 0.02 | 20 |
| B | 0.01 | 20 |





| ranks |      |     |
|-------|------|-----|
| A     | 0.99 | 1   |
| A     | 0.91 | 2.5 |
| A     | 0.91 | 2.5 |
| A     | 0.64 | 5.5 |
| A     | 0.64 | 5.5 |
| A     | 0.27 | 8.5 |
| A     | 0.16 | 12  |
| A     | 0.16 | 12  |
| A     | 0.16 | 12  |
| A     | 0.03 | 17  |
| B     | 0.64 | 5.5 |
| B     | 0.64 | 5.5 |
| B     | 0.27 | 8.5 |
| B     | 0.16 | 12  |
| B     | 0.16 | 12  |
| B     | 0.08 | 15  |
| B     | 0.03 | 17  |
| B     | 0.03 | 17  |
| B     | 0.02 | 19  |
| B     | 0.01 | 20  |

Perform  $t$  test on Ranks

|        | $A_{\text{rank}}$ | $B_{\text{rank}}$ |
|--------|-------------------|-------------------|
| avg    | 7.85              | 13.15             |
| stdDev | 5.28              | 5.33              |

|                                                      | Ranked $t$ Test |             |
|------------------------------------------------------|-----------------|-------------|
| $s_T = \sqrt{\frac{s_A^2}{n_A} + \frac{s_B^2}{n_B}}$ | 2.37            | $n = 10$    |
| $(\text{avg}_A - \text{avg}_B) / s_T$                | 2.23            | $t_R$ score |
| $p$ -value                                           | 0.038           |             |

Ranked Example



## Ranked $t$ Test: What do we pay?

- $t$  Test is optimized for the normal distribution
- $t$  Test on the ranks is not
  - How much do we pay?

| Distribution | # Samples for $t$ Test | # Samples for $t$ Test on Ranks | # Samples of $t_R$ , normalized to 50 runs of $t$ |
|--------------|------------------------|---------------------------------|---------------------------------------------------|
| Normal       | 31                     | 32                              | 52                                                |
| Exponential  | 29                     | 16                              | 27                                                |
| Uniform      | 31                     | 34                              | 55                                                |
| Bimodal      | 31                     | 34                              | 54                                                |
| Chubby Tails | 40                     | 12                              | 15                                                |



## A Non-Parametric ‘Mean’: The Median

- Average of a data set that is not normally distributed produces a value that behaves non-intuitively
  - Especially if the probability distribution is skewed
    - Large values in ‘tail’ can dominate
    - Average tends to reflect the typical value of the “worst” data not the typical value of the data in general
- Instead use the Median
  - 50<sup>th</sup> percentile
  - Counting from 1, it is the value in the  $\frac{n+1}{2}$  position
    - If  $n$  is even,  $(n+1)/2$  will be between 2 positions, average the values at that position



## A Confidence Interval Around the Median: Thompson-Savur

- Find the  $b$  the binomial value that has a cumulative upper tail probability of  $\alpha/2$ 
  - $b$  will have a value near  $n/2$
- The lower percentile  $l = \frac{b}{n-1}$
- The upper percentile  $u = 1 - l$
- Confidence Interval is  $[value_b, value_u]$ 
  - i.e.  $value_l \leq median \leq value_u$
  - With a confidence level of  $1 - \alpha$



## A Confidence Interval Around the Median: Thompson-Savur

- In Excel:
  - To calculate  $b$  use  
`CRITBINOM (n, 1/2,  $\alpha/2$ )`
  - to compute the  $value_u$  use the function  
`PERCENTILE (dataArray, u)`
  - to compute the  $value_l$  use the function  
`PERCENTILE (dataArray, l)`



## A Confidence Interval Alternative to the Ranked $t$ Test

- Find the median confidence interval for the two data sets
- If the confidence intervals do not overlap
  - Data sets are taken from different distributions
  - With a confidence level of  $1 - \alpha$  where  $\alpha$  is the upper tail probability used in computing  $b$
  - Advantages:
    - Gives better understanding of system
      - see median values with error bounds
    - Easy to draw and productive on a graph
  - Disadvantage:
    - Not as sensitive as the ranked  $t$  test



## Does My Difference Matter?

- Okay, so your results are significantly better than the published results. So what?
  - Statistics can answer, “is it better?”, but not “does it matter?”
- You perform 100 000 runs of your classifier and 100 000 runs of the reference classifier
  - You get a  $t$  score of 31.6! ☺
  - The  $p$ -score is reported by Excel as 0! (Actually  $2.0 \times 10^{-219}$ )
  - But...your way classifies data at 91.0% accuracy, whereas the reference technique classifies at 90.8% accuracy.
  - Not much difference!
    - Especially if your technique is much slower than the reference way



## Measuring Effect Size

- One statistic for effect size: Cohen's  $d'$ 
  - $d'$  is computed by  $d' = \frac{t}{\sqrt{(n_1 + n_2)/2}}$
  - Measures the difference between means in terms of the pooled standard deviation
  - Cohen suggests that 0.25 is a small difference; 0.50 is a medium-sized difference; 0.75 is a large difference
  - For our example,  $d'$  is 0.10
    - Essentially an insignificant difference
- Problem: we did too many runs!



## Perils of Computer Science

- Very easy to do a host of runs
  - 1 000 000 runs of each group not too hard on a PC
    - $t$ -scores around 500
    - Bizarre-looking results
      - $p$  scores show up as 0 or #NUM! in Excel
- We often know that there *will* be a difference between our method and the reference method
  - Enough runs will detect this difference
  - Difference might only be slight
    - Use care in choosing sample sizes, or (*better*) compute Cohen's  $d$



## Perils of Computer Science

- We can generate lots of data very quickly
  - Leads to over-complicated experimental designs
  - Many parameters in Evolutionary Computation
    - 10 levels of A, 10 levels of B, 10 levels of C = 1000 treatments!
    - Tough to analyze
    - A simple  $t$  test on the ranks is nearly always better (and see following)
- Always draw a scatter plot or histogram of your data!
  - This alerts you to strange things, like that the mean is very bad, but some individuals are very good
- Always record the performance of *ALL* the individuals
  - You'll need this for doing the  $t$  test on the ranks
  - In EC, we mean *ALL* individuals of *interest*; i.e. best of run



## Perils of Computer Science in Evolutionary Computation

- Don't confuse Population averages with Best-of-Run averages!
  - In any GA or GP, the average of the population tells you almost nothing of interest
  - Use the median of the best-of-run, do the *WHOLE* experiment several times
  - Use the tree size of the best-of-run individuals as well!
    - They are the Heroes – hence they are of interest, unless you're really looking to optimize average tree size during evolution
- Test 2 (or a few) parameter settings, one variable at a time
  - Fewer runs required
  - Get a feel for how the variation occurs



## Repetitions

- What is the number of repetitions needed to see if there is a difference between two means or between two medians?
  - Depends on the underlying distributions
    - But underlying distributions are unknown
- Rule of thumb
  - Perform a minimum of 30 repetitions for each system
  - Performing 50 to 100 repetitions is usually better



## More Than 2 Treatments

- Preceding stats to be used for simple experiment designs
- More sophisticated stats needs to be done if:
  - Comparing multiple systems instead of just 2 treatments
    - E.g. comparing the effect on a Genetic Algorithm of using no mutation, low, medium and high levels of mutation
      - We say there are 4 *levels* of the mutation variable
      - Need  $\binom{4}{2} = 6$  possible comparisons to test all pairs of treatments
  - Called a 'multi-level' analysis



## Multiple Levels: Post-hoc Analysis

- For 4 levels of mutation there are 6 comparisons possible
  - *Each one* of the comparison holds at a 95% C.L. independent of the other comparisons
  - If *all* comparisons are to hold at once the odds are  $0.95 \times 0.95 \times 0.95 \times \dots \times 0.95 = (0.95)^6 = 0.735$
  - So in practice we only have 73.5% C.L.
    - Wrong 1/4 of the time
- For 7 levels of mutation there are 21 comparisons possible
  - C.L. =  $(0.95)^{21} = 0.341$ 
    - Chances are better than half that at least one of the decisions may be wrong!



## The Bonferroni Correction for Tests

- To correct, choose a smaller  $\alpha$ 
$$\alpha' = \frac{\alpha}{m}$$
  - Where  $m$  is the number of comparisons
  - So for 95% CL use  $\alpha = 0.025/6 = 0.004167$
  - For a Z test the critical value changes from 1.96 to 2.64
- Called a Bonferroni post-hoc correction
  - Other post-hoc techniques such as Tukey and Scheffé that are more powerful than Bonferroni; also Holm's and Sidak's procedures can be useful
- You should apply the Bonferroni correction:
  - To  $t$  tests ( $t$  tests and ranked  $t$  tests)
  - To Confidence Intervals and Error Bounds
  - Whenever you mean "all the significant results we found hold at once"



## The Bonferroni Correction for Experiments

- The Bonferroni Correction is more widely applicable than just for multi-level comparisons
- We really need to control for the dilution of the confidence levels throughout the study, whether or not the CLs are applied to analyses of independent 'phenomena'
  - We must *divide* the  $\alpha$  used for each CL test by the total number of CL tests in the study
- To apply the Bonferroni correction to  $p$ -values *multiply* the  $p$ -values by the number of CL tests performed
  - "Probabilities" bigger than 1 means "not significant"



## The Bonferroni Correction for Experiments

- Example:
  - A robot dog has been created
    - Genetic Programming is used to control the ear wiggles of the robot
    - a Genetic Algorithm is used to optimize its tail wagging ability
  - A study is being done to improve both the ears and the tail independently, and we want to be 95% confident in our over all tests
    - For the ears the GP is tested with 3 different sets of terminal nodes
    - For the tail the GA is tested with 4 different fitness functions
    - There are  $\binom{3}{2} + \binom{4}{2} = 3 + 6 = 9$  total CL inferences used in the study
    - Consequently the  $\alpha$  used for any CL should be  $\alpha = 0.025 / 9 = 0.0028$



## Multiple Factors

- Most of the time, there are many different properties we are interested in studying
  - e.g. We may be trying out various kinds of crossovers, with and without mutation, under different selection pressures
  - Each of the above parameters has multiple levels
  - This is called a multiple factor analysis
    - with each factor having multiple levels
  - Use Analysis of Variance or General Linear Models to analyze
    - See text books on ANOVA and GLMs



## Multiple Factors: Factorial Design

- When dealing with multiple factors with multiple levels
  - Important that all combinations of factor levels are tried
  - A given combination of factor levels is called a treatment
  - If you want accurate information about each possible interaction, each treatment should be repeated at least 30 times
    - If you interested largely in main effects, 10 repetitions is often fine, if you have enough levels



## Multiple Factors: Factorial Design

E.g. if we have 2 EC systems, new and standard (New and Std) and we want to see their behavior under

- crossover and no crossover (x and ✖)
- 3 different selection pressures (p1, p2 and p3)

|   | t1  | t2  | t3  | t4  | t5  | t6  | t7  | t8  | t9  | t10 | t11 | t12 |
|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| S | New | New | New | New | New | New | Std | Std | Std | Std | Std | Std |
| X | x   | x   | x   | ✖   | ✖   | ✖   | x   | x   | x   | ✖   | ✖   | ✖   |
| P | p1  | p2  | p3  | p1  | p2  | p3  | p1  | p2  | p3  | p1  | p2  | p3  |



## Multiple Factors: Factorial Design

- If we are performing 50 reps per treatment
  - In previous example we have
$$S \times X \times P \times 50 = 2 \times 2 \times 3 \times 50 = 12 \times 50 = 600 \text{ experiments to perform}$$
- The number of experiments goes up as the product of the number of levels in each factor
  - This is exponential in the number of factors
  - Consequently, carefully choose the factors and factor levels that you study in your experiments
  - Minimize what factors you vary (focus your experiments on the relevant factors)



## Top 12 Statistics Myths in EC

1. My mean result being better than yours means my technique is superior to yours
  - In the best case you need to perform a  $t$  test to assert this claim
2. Reporting the mean value of a statistic is good enough
  - You need some representative range
3. Reporting the mean and standard deviation of a statistic is good enough
  - Need number of runs
4. Your data are normally distributed
  - Not usually



## Top 12 Statistics Myths in EC

5. The mean performance of the best-of-run individuals of your system is what matters
  - It's usually the median you want
6. 10 runs is enough to show significant differences between groups
  - It can be, but the statistics required to show this are hairy
7. 95% confidence levels are generally sufficient
  - Try 99.9%
8. Drawing 95% confidence intervals around each sample mean on a graph implies that it's a rare event if any of the true means fall outside the CIs
  - Nope; need Bonferroni correction



## Top 12 Statistics Myths in EC

9. Reporting the results of several comparisons where each is made at a 95% confidence level means that all conclusions are valid simultaneously
  - Nope; need Bonferroni correction for that too
10. 95% confidence intervals can be computed using the sample mean  $\pm 1.96$  standard deviations of the mean
  - Nope; need the Student's  $t$  score given your degrees of freedom
11. An experimental setup where more than one parameter is varied can be treated like one where exactly one parameter varies
  - Need ANOVA, MANOVA or regression
12. One can infer trends from observed data beyond the data you've generated
  - Generally, this would be a consequence of some model, and you probably haven't supported said model with enough experimental data



## References

- Slides online:  
<http://www.scs.carleton.ca/~schrste/tamale/UsingAppropriateStatistics.pdf>
- Hyperstat Online Textbook:
  - <http://davidmlane.com/hyperstat/index.html>
  - Statistics textbook for psychology students
    - Easy math, nice examples. ☺
- Statistics Chapter of Numerical Recipes in C
  - <http://www.library.cornell.edu/nr/cbookcpdf.html>
  - Chapter 14, “Statistical Description of Data”
  - Very detailed, more for advanced users