Noise-Induced Features in Robust Multi-Objective Optimization Problems

C. K. Goh, K. C. Tan, C. Y. Cheong and Y. S. Ong

Abstract—Apart from the need to satisfy several competing objectives, many real-world applications are also sensitive to decision or environmental parameter variation which results in large or unacceptable performance variation. While evolutionary optimization techniques have several advantages over operational research methods for robust optimization, it is rarely studied by the evolutionary multi-objective (MO) optimization community. This paper addresses the issue of robust MO optimization by presenting a robust continuous MO test suite with features of noise-induced solution space, fitness landscape and decision space variation. The work presented in this paper should encourage further studies and the development of more effective algorithms for robust MO optimization.

Index Terms—Evolutionary algorithms, multi-objective optimization, robust solutions, robust test functions

I. INTRODUCTION

While the application of evolutionary multi-objective optimization (EMOO) to real-world applications has been gaining significant attention from researchers in different fields, there is a distinct lack of studies investigating the issues of uncertainties in the literature [6]. Uncertainties include different factors such as data incompleteness, mathematical model inaccuracies, environmental condition variation, and solutions that cannot be implemented exactly. This paper focuses on the optimization of robust solutions that remain satisfactory in the face of parametric variations for MO problems. The optimization of robust solutions is of particular importance in real-world problems, such as scheduling, vehicle routing and engineering design optimization, where certain characteristics of the environment may not be known with absolute certainty and the decision spaces can be sensitive to parametric variations.

Variation in design variables or the environment may affect the quality and performance adversely and robust optimization considers the effects explicitly and seeks to minimize the consequences without eliminating efficiency. In operational research, robust optimization is considered as a modeling methodology where the robust problems are reformulated into the form of linear, conic quadratic and semidefinite programming problems. Nonetheless, assumptions or approximations are often made during problem reformulation to ensure computational tractability resulting in more uncertainties in the problem model. In addition, it does not allow for the incorporation of any domain knowledge to achieve better performance. Other approaches including Taguchi orthogonal arrays, response surface methodology, probabilistic design analysis, and etc. have also been applied for robust optimization. On the other hand, evolutionary optimization techniques do not have such limitations, making it appropriate for robust optimization.

A number of studies concerning evolutionary single-objective (SO) optimization of robust solutions have been reported in the literature [1], [2], [11], [14]. Nonetheless, robust MO problems are rarely considered in the literature until recently [3], [8]. For example, Deb and Gupta [3] considered the optimization of the mean objective values as well as the formulation of the robust MO problem as a constrained problem. In contrast to SO optimization, it is also essential to obtain a well-distributed and diverse solution set in MO optimization. Although the general concepts of existing evolutionary robust SO techniques can be easily extended for robust MO optimization, the robust optimization process is complicated by the need to consider several issues that are unique to MO optimization explicitly. Therefore, the challenge in the design of any robust MOEA is to deal with issues such as the notion of Pareto-optimality, elitism and balance between exploration and exploitation with the additional consideration of robustness.

This paper examines the suitability of existing robust test problems for MO optimization and presents a set of guidelines for the construction of robust MO test problems. The fundamental component of the robust test problems is a Gaussian landscape generator that facilitates the specification of robust optimization-specific features such as noise-induced solution space, fitness landscape and decision space variation. This generator is developed with the purpose of generating noise-sensitive landscapes in conjunction with existing MO test problems, and due to its independent nature, it can be used to generate robust SO test problems as well. Subsequently, a robust MO test suite is built upon the ZDT framework.

The organization of the paper is as follows: Section II provides some background information on the concepts of robust MO optimization while a categorization of robust MO problems is presented in Section III. Subsequently, in Section IV, the robust Gaussian landscape generator and the robust MO test construction guidelines are presented, together with the continuous test suite. Conclusions are drawn in Section V.

II. ROBUST MO OPTIMIZATION

In order to reduce the consequences of uncertainty on optimality and practicality of the solution set, factors such as...
as decision variable variation, environmental variation and modeling uncertainty have to be considered explicitly. Therefore, the minimization robust MO problem is defined as the following.

\[
\min_{\vec{x} \in \vec{X}^*} \tilde{f}(\vec{x}, \sigma^x, \sigma^e) = \{ f_1(\vec{x}, \sigma^x, \sigma^e), f_2(\vec{x}, \sigma^x, \sigma^e), \ldots, f_M(\vec{x}, \sigma^x, \sigma^e) \}
\]

\text{s.t. } \tilde{g}(\vec{x}, \sigma^x, \sigma^e) \geq 0, \tilde{h}(\vec{x}, \sigma^x, \sigma^e) = 0
\]

where \(\sigma^x\) and \(\sigma^e\) represent, respectively, the uncertainties associated with \(\vec{x}\) and environmental conditions. Both forms of uncertainties may be treated equivalently.

In order to avoid any confusion in the subsequent discussions, it will be instructive to make a distinction between the notations used for deterministic MO and robust MO optimization. The terms \(\text{PF}^*\) and \(\text{PS}^*\) refer to the desired Pareto front and solution set in the general sense, without representing any specific case. The optimal Pareto front and the corresponding Pareto solution set of a particular deterministic MO problem will be denoted as \(\text{PF}_{\text{det}}^*\) and \(\text{PS}_{\text{det}}^*\) respectively. Note that \(\text{PF}_{\text{det}}^*\) may not be known \textit{a priori} and it is fixed for any particular MO problem.

In the case of robust MO optimization, the optimal robust Pareto front and solution set are dependent on the noise model and the robust measure. This implies that, contrary to \(\text{PF}^*\) and \(\text{PS}^*\), the optimal robust Pareto set is not fixed. Furthermore, the structure of the Pareto front, i.e. its dimensionality may change as well due to the additional optimization criterion of robustness. Therefore, the notation should reflect the noise model and the robust measure used. In this paper, the optimal robust Pareto front and optimal solution set are denoted as \(\text{PF}_{\text{rob}}^*\) and \(\text{PS}_{\text{rob}}^*\) respectively. The terms \(\text{rm}\) and \(\text{\sigma}\) refers to the robust measure and noise model in consideration respectively. Accordingly, \(\text{PF}_{\text{rob}}^*\) refers to the final set of nondominated solutions evolved by robust MOEA based on the robust measure, \(\text{rm}\) and noise model, \(\sigma\).

III. ROBUST OPTIMIZATION PROBLEMS

This section presents a set of guidelines for the construction of robust MO test problems. Based on the existing literature on robust optimization, Section III-A reviews the different categorization of robust problems and presents a classification scheme applicable to MO optimization. Subsequently, the robust landscape generator and detailed construction guidelines are presented in Section III-B.

A. Robust MO Problem Categorization

Different categorization of robust problems have been considered in the literature. Jin and Branke [9] states that robust optimization can be considered from the perspective of solution sensitivity to decision variable variations or environmental variations. Decision variable variations stem from the fact that deviations from design specifications are inevitable in manufacturing. On the other hand, environmental variable variations arise from variations in operational or environmental conditions.

Paenke et al [12] proposed four categories based on the relationship between the efficient and effective fitness landscapes: 1) identical optimum where efficient and robust optima are identical, 2) neighborhood optimum where efficient and robust optima are located on the same peak or trough, 3) local-global flip where one of the local optima corresponds to the robust optimum, and 4) max-min flip where the global maximum corresponds to the robust optimum. Deb and Gupta [3] considered a similar classification that is specific to the context of MO optimization: 1) the global efficient front is robust, 2) a part of the global efficient front is not robust, 3) the robust front is represented by a local efficient front, and 4) the robust front is represented by both the global and local efficient fronts.

Robust MO problems are certainly much more complex than SO problems because both decision and objective space are susceptible to change due to uncertainties. Recent studies [1], [13] demonstrated that some problems have the interesting property of demonstrating fitness topological changes in the presence of noise. To be precise, topological variation strictly refers to the introduction of new problem features to the deterministic problem under the influence of noise. For the two classification schemes described above, problems of the first category are typically considered to be less interesting as compared to problems of the other classes. On the other hand, it is possible that noise-induced landscape variations can actually result in a more challenging optimization problem even if the location of the optimum remains the same. Moreover, a landscape transformation may result from the addition of different robust criteria as objectives to be solved. Therefore, it will certainly be more interesting to classify robust MO problems according to the aspects of change under the influence of noise.

Most benchmark problems in the literature are commonly characterized by the emergence of a local optimum as the most robust solution in the presence of noise, signifying a change in the location of the optimum, and in the context of MO problem, a change in \(\text{PS}^*\). Moreover, as mentioned above, it is distinctly possible that the whole fitness topology changes leading to two distinct types of search space variation. As noted by Deb and Gupta [3], the \(\text{PF}^*\) is also susceptible to changes.

For the classification of robust MO problems, this paper presents a three-bit binary classification scheme. In this scheme, the bits represents the presence of \(\text{PF}^*, \text{PS}^*\), and landscape changes respectively, in decreasing significance. Therefore, there is a total of eight classes. At one end, we have a class 0 problem representing a problem that does not change under the influence of noise while the other end represents a problem which exhibits all three types of changes. As a specific instance, a MO problem that demonstrates landscape and \(\text{PS}^*\) changes is a class 3 problem under this classification.

Several desirable properties of deterministic benchmarks and test suites have been suggested in the EA literature. In addition to these guidelines, the following issues should be considered in the development of robust benchmark problems in the context of MO optimization:
• Robust MO problems are essentially MO problems and guidelines for the construction of MO benchmark problems established in the existing literature should be taken into account;
• The PF* of the test functions should not be any more difficult to find compared to PF* when conventional MOEAs are applied;
• Some test problems should contain existing or emergent features that pose more difficulties when robust MO optimization techniques are employed;
• The “sensitive” component of the benchmark problems should be scalable;
• Some test problems should contain possible tradeoff in robustness between different objectives.

In general, any test function should be simple enough to allow for analysis of algorithmic behavior but, at the same time, complex enough to allow conjectures to the real-world. However, a quick survey of past works will reveal the lack of problem characteristics beyond the basic landscape featuring contrastive sharp and broad peaks or troughs in the evaluation of uncertainty-handling techniques. In particular, some robust SO test functions may be too simplistic for proper algorithmic evaluation with the apparent lack of difficulties that may hinder the selection of robust MO solutions. Furthermore, some robust benchmarks are distinctly multi-modal in nature and it may be difficult to ascertain whether the robust solution found is the consequence of premature convergence or the effectiveness of the particular robust optimization technique.

B. Robust MO Test Problem Design

The fundamental component of the robust MO test functions proposed in this paper is a Gaussian landscape generator that introduces various parametric sensitivities to the deterministic fitness landscape. It generates a set of \( n_x \)-dimensional minima throughout the fitness landscape and is given by:

\[
b(x_r) = 1 - \frac{1}{|x_r|} \sum_{i \in x_r} \max_{j \in J} \left\{ h_{ij} \exp\left[\frac{(x_i - \mu_{ij}) \cdot E_{ij}(\sigma, s_{ij})}{w}\right] \right\}
\]

\[
E_{ij}(\sigma, s_{ij}) = 1 + s_{ij} \cdot U(-\sigma, \sigma)
\]

where \( J \) is the number of basis functions, \( d_j, \mu_{ij}, \) and \( w \) denote the amplitude, location and the width of the basis functions respectively. \( E_{ij} \) controls how the environmental variable behaves with noise, \( \sigma \) and the degree of sensitivity, \( s_{ij}. \) Intuitively, the robustness of a particular basin will depend on the associated \( E_{ij} \) function, while the amplitude will determine the optimality of the solution. From Eqs 2 and 3, it can be noted that test functions designed using this landscape generator are different from most previous works in two aspects:

• Any solution space or objective space transformation is a consequence of environmental variation. Although environmental parameter variation is rarely considered in the literature, it is definitely more flexible compared to decision parameter variation when it comes to the design of different possible scenarios.

• As observed from the simulation studies conducted in the previous section, it is important for the basin of attraction of the various troughs to be very similar. This ensures that there is no initialization bias towards any particular region of the search space.

The max function has been used successfully in previous works [5], [7] to combine the different Gaussian components and it ensures that the landscape feature at any one point is determined and influenced only by the dominant basin. Without the overlapping influences from the other basis functions, this allows each basis function to be considered independently and facilitates the design and analysis of the robust test function. In particular, it is possible to define explicitly the location and depth of the different basins to create different test functions with specific characteristics. For the purpose of evaluating algorithmic performance, it is necessary to know the relative degree of robustness for each minimum. The theoretical values for each basis function can be easily worked to be:

\[
B_j = d_j \cdot \left( \frac{w \sqrt{\pi}}{2s_{ij} \sigma} \right) \cdot e^{r f\left( \frac{s_{ij} \sigma}{w} \right)}
\]

In this paper, the robust MO test problems are built upon the ZDT framework. The flexibility of this framework has also been demonstrated by the development of a suite of dynamic MO problems by Farina et al in [4]. The guidelines for the construction of the deterministic ZDT test functions are formally described by the following

\[
\begin{align*}
\min f_1(x_{d1}) &= x_1 \\
\min f_2(x_{d2}) &= g(x_{d2}) \cdot h(f_1, g)
\end{align*}
\]

where \( x_{d1}, x_{d2} \in \bar{x}, \) and the \( g \) and \( h \) functions control the problem difficulty and the shape of the Pareto front, respectively. For the ensuring discussions, we assume that the particular ZDT problem to be extended has the following functional form,

\[
\begin{align*}
g(x_{d2}) &= 1 + \sum_{x \in x_{d2}} x_i \\
h(f_1, g) &= 1 - \left( \frac{f_1}{F_1} \right)^7.
\end{align*}
\]

1) Basic landscape generation: Noise-induced changes to the PF*, PS* and fitness landscape can be introduced by incorporating \( b \) into either the \( h \) and/or \( g \) functions to construct different classes of robust test problems. A straight-forward approach of introducing robust features into the problem is to change \( g \) in the form of \( g(\bar{x}) = 1 + b(x_r), \) with \( h \) and \( f_1 \) unchanged. \( \bar{x} \) is also a subset of \( \bar{x}. \) Let us consider a two-dimensional landscape generated by

\[
b(x_r) = 1 - \frac{1}{|x_r|} \sum_{i \in x_r} \max\left\{ 0.8 \exp\left[\frac{(x_i - 0.25E_{ij}(\sigma, s_{ij}))^2}{0.4} \right], \exp\left[\frac{(x_i - 0.75E_{ij}(\sigma, s_{ij}))^2}{0.4} \right] \right\}
\]

The problem landscape presented by \( b \) at \( \sigma = \{0, 0.15\}, \) and the resulting Pareto fronts are shown in Fig. 1. The minimum located at \( (0.75, 0.75) \) is the global minimum in a deterministic setting and failure to converge to this point will result in a dominated solution. With \( s = 1, \) notice that the effect of noise on the basin at \( (0.75, 0.75) \) is actually three times more than that on the basin at \( (0.25, 0.25). \) Thus, when
noise is incorporated into the problem, the local minimum at (0.25,0.25) constitutes to the PF*_{trim,n} as it is more robust and its performance is less affected by noise. Since the only change induced by noise is the location of PS*, this is a simple instance of a class 2 robust problem.

On the other hand, if the g and b functions are combined such that

\[
g(\vec{x}) = (1 + \sum_{x_1 \in x_2} x_i) + b(\vec{x}),
\]

(8)

the resulting problem is also a class 2 problem. However, such a formulation allows the analysis of the effects of robust optimization on the original problem. Thus, the robust MOEA must be capable of finding the global robust minima associated with b as well as dealing with the difficulties posed by the deterministic problem in order to find PF*_{trim,n}. It is also possible to redefine \( f_1 \) as \( f_1(\vec{x}) = x_1 + b(\vec{x}) \) to construct a class 2 problem with similar properties.

2) Changing the decision space: When combined with \( g \) in the ways described above, the \( b \) function gives rise to the element of noise-induced changes to the PS* and results in class 2 test problems. Features of noise-induced search space variation can be easily incorporated into the problem by changing \( g \) in the following form,

\[
g(\vec{x}) = 1 + \left( \sum_{x_1 \in x_2} x_i \right) \beta - h(\vec{x}),
\]

(9)

which forces the distribution of the solutions to change. Notice that \( g \) is now a function of \( b \). In this particular instance, it is possible to apply Eqn. 7 as the \( b \) function but finding its minimum will have no direct contribution to solution optimality. Interestingly, finding the optimal value for \( b \) will improve the distribution of the solutions near PS* and hence, simplify the problem somewhat. Thus, the resulting problem can be considered as a class 1 test problem.

More complex fitness topology variations can be induced by making \( h \) a function of \( g \) instead. In particular, consider the scenario where we define \( b \) such that the width, i.e., the size of the basin of attraction, of the selected minimum is a function of \( g \) and replace the \( g \) function by

\[
g'(\vec{x}) = g(x_{1d}) + b(\vec{x}),
\]

(10)

The corresponding problem depends on the characteristics of the \( b \) function; it is a class 1 test problem if \( J = 1 \) and class 3 test problem if \( J \geq 1 \) and deterministic and robust optima are different. In any event, the robust MOEA must be able to deal with the features that arise due to noise in order to find PF*_{trim,n}.

3) Changing the solution space: Since the shape of the PF*_{det} is determined by the \( h \) function in the ZDT framework, PF*_{trim,n} can be easily controlled by combining the \( b \) and \( h \) in some appropriate ways. The simplest way to introduce changes in PF* is to control its convexity:

\[
h(f_1, g, x) = 1 - \left( \frac{f_1}{g} \right)^{\alpha + h(x)}.
\]

(11)

If the \( g \) function is unchanged and \( b \) defines a single basin, only the convexity of the PF* is affected by noise while
PS* remains the same. Thus, the resulting problem is a class 4 test problem and the robust MOEA must be capable of distributing the solutions along PF* with varying noise-induced convexity. However, if \( b \) is characterized by multiple basins as illustrated in Fig. 1, both the PF* and PS* will change and the corresponding problem becomes a class 6 test problem instead.

It is also possible to redefine the \( h \) function as,

\[
h(f_1, g, b) = b(x_r, f_1) - \sqrt{\frac{f_1}{g}}.
\] (12)

where \( b \) is now a function of \( f_1 \) as well. One interesting implication of such a formulation, particularly if the sensitivities of the relevant basins increase with \( f_1 \), is the resulting tradeoff between the robustness and optimality of \( f_2 \). Therefore, a part of the PF* will become dominated in the presence of noise and hence only part of \( x_1 \) makes up the PF_{rob,n}. Intuitively, the corresponding problem is a class 6 test problem.

4) Example of a robust MO test suite: Having described the possible modifications to extend the ZDT test problems, we are now in the position to suggest a suite of five robust MO test problems that satisfy the requirements described previously. Although not all seven classes of problems are
TABLE I
DEFINITIONS OF THE GTCO TEST SUITE

<table>
<thead>
<tr>
<th>Problem</th>
<th>Type</th>
<th>Definition</th>
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| GTCO1   | Class 1 | $f_1(x_{d1}) = x_1$  
$g(x_{d2}) = 1 + \sum_{i=2}^{d2} \{ (x_i - 0.5)^2 + b(x_i, x_r) \}$  
$h(f_1, g) = 1 - \sqrt{\frac{4}{g}}$  
$b(x_r, x_i) = 1 - \frac{1}{|x_r|} \sum_{j \in x_r} \exp \left[ \frac{(x_r - W(x_j))^2}{W(x_j)} \right]$  
$W(x_i) = 0.1 + 0.1 \cos(20(x_i - 0.5) \pi) \cdot (1 - |x_i - 0.5|)^5$  
$E_{ij}(\sigma, s) = U(-\sigma, \sigma), \ x_{d1}, x_{d2}, x_r \in [0, 1]$ |
| GTCO2   | Class 2 | $f_1(x_{d1}) = x_1$  
$g(\bar{x}) = 1 + b(x_r)$  
$h(f_1, g) = 1 - \sqrt{\frac{4}{g}}$  
$b(x_r) = 1 - \frac{1}{|x_r|} \sum_{i \in x_r} \max \left\{ 0.8 \exp \left[ \frac{(x_i - 0.25E_1(\sigma, s))^2}{0.1} \right], \exp \left[ \frac{(x_i - 0.75E_2(\sigma, s))^2}{0.1} \right] \right\}$  
$E_{i1}(\sigma, s) = 1 + U(-\sigma, \sigma), \ x_{d1}, x_r \in [0, 1]$ |
| GTCO3   | Class 3 | $f_1(x_{d1}) = x_1$  
$g(\bar{x}) = 1 + 10(\sum_{i=2}^{d2} \frac{|x_i|}{|x_i| - 1})^{1.25} - b_1(x_{r1}) + b_2(x_{r2})$  
$h(f_1, g) = 1 - \sqrt{\frac{4}{g}}$  
$b_1(x_{r1}) = 1 - \frac{1}{|x_r|} \sum_{i \in x_r} \exp \left[ \frac{(x_i - 0.25E_{i1}(\sigma, s))^2}{0.05} \right]$  
$b_2(x_{r2}) = 1 - \frac{1}{|x_r|} \sum_{i \in x_r} \max \left\{ 0.8 \exp \left[ \frac{(x_i - 0.25E_{i2}(\sigma, s))^2}{0.1} \right], \exp \left[ \frac{(x_i - 0.75E_{i2}(\sigma, s))^2}{0.1} \right] \right\}$  
$E_{i1}(\sigma, s), E_{i2}(\sigma, s) = 1 + U(-\sigma, \sigma), \ x_{d1}, x_{d2}, x_{r1}, x_{r2} \in [0, 1]$ |
| GTCO4   | Class 6 | $f_1(x_{d1}) = x_1$  
$g(x_{d2}) = 1 + 10 \sum_{i=2}^{d2} x_i$  
$h(f_1, g, x_r) = 1 - \left( \frac{4}{g} \right)^{1/2}$  
$\alpha = 0.5 + b(x_r)$  
$b(x_r) = 1 - \frac{1}{|x_r|} \sum_{i \in x_r} \max \left\{ 0.8 \exp \left[ \frac{(x_i - 0.25E_{i1}(\sigma, s))^2}{0.05} \right], \exp \left[ \frac{(x_i - 0.75E_{i1}(\sigma, s))^2}{0.05} \right] \right\}$  
$E_{i1}(\sigma, s) = 1 + U(-\sigma, \sigma), \ x_{d1}, x_{d2}, x_r \in [0, 1]$ |
| GTCO5   | Class 7 | $f_1(x_{d1}) = x_1$  
$g(\bar{x}) = 1 + \sum_{i=2}^{d2} \{ x_i + 5b_1(x_{r1}) \}$  
$h(f_1, g) = 1 + 2b_2(x_{r2}, f_1) - \sqrt{\frac{4}{g}}$  
$b_1(x_{r1}, x_i) = 1 - \frac{1}{|x_r|} \sum_{j \in x_r} \frac{(x_j - W(x_i))^2}{W(x_i)}$  
$b_2(x_{r2}, f_1) = 1 - \frac{1}{|x_r|^2} \sum_{j \in x_r} \exp \left[ \frac{(x_j - W(x_i))^2}{W(x_i)^2} \right]$  
$W_1(x_i) = \begin{cases} 0.2, & \text{if } x_i < 0.05 \\ 0.1x_i + 0.05, & \text{otherwise} \end{cases}$  
$W_2(f_1) = 0.2 \cdot (1 - \sqrt{f_1})$  
$E_{i1}(\sigma, s) = U(-\sigma, \sigma), \ x_{d1}, x_{d2}, x_{r1}, x_{r2} \in [0, 1]$ |
represented, these problems embody the most challenging aspects of robust MO optimization that have been described previously. Nonetheless, interested readers are encouraged to construct more interesting problems based on the guidelines made in the previous sections. At this point, it is worth mentioning that the proposed \( b \) function can also be employed as a non-optimizable component of the problem and as a noise-sensitive environment variable instead, i.e. \( b(R) \). The definitions of the test suite are summarized in Table I.

GTCO1 utilizes the effects of Eqn. 10 to bring about a change from unimodal at \( \sigma = 0.0 \) to multimodal fitness landscape at \( \sigma = 0.2 \) as shown in Fig. 2. The \( \text{PS}_{\text{det}} \) and \( \text{PS}_{\text{rm,n}} \) are the same at all noise levels and correspond to \( x_i \in x_{d0} = 0 \) and \( x_i \in x_{r} = 0 \). The problem becomes increasingly multimodal with increasing \( \sigma \) and this is an instance where the problem becomes more challenging and the robust MOEA will face difficulties finding \( \text{PF}^* \) due to the landscape change. The settings of \( |x_{d0}^*| = 10, |x_r^*| = 5 \) and \( \sigma = 0.2 \) are recommended for GTCO1.

GTCO2 is an instantiation of the two-minima scenario considered in Section III-B.1. This problem is similar to the problem of rMOP4 in the sense that the deterministic global and local minima are switched when noise is increased beyond a threshold as shown in Fig. 3. However, as mentioned before, the basins of attraction for both minima are the same, eliminating initialization bias. The \( \text{PS}_{\text{det}}^* \) corresponds to \( x_r^* = 0.75 \) while \( \text{PS}_{\text{rm,n}}^* \) corresponds to \( x_r^* = 0.25 \). The settings of \( |x_{d0}^*| = 10 \) and \( \sigma = 0.2 \) are recommended for GTCO2.

GTCO3 represents a combination of GTCO2 and the effects of Eqn. 9 to induce both fitness landscape and \( \text{PS}^* \) changes in the presence of noise. Noise-induced changes to the decision space are similar to GTCO2 except that the density of the Pareto optimal solutions is now adversely affected by noise. The behaviors of the solution space at \( \sigma = 0.0 \) and \( \sigma = 0.2 \) are shown in Fig. 4, where it can be noted that the bias away from the \( \text{PS}^* \) will be attenuated with increasing \( \sigma \) values. The density of Pareto optimal solutions is at its highest and, hence easiest to find, when \( x_{d0}^* = 0.0 \). The settings of \( |x_{d0}^*| = 10, |x_{r1}^*| = |x_{r2}^*| = 5 \) and \( \sigma = 0.2 \) are recommended for this problem.

Noise-induced changes in \( \text{PS}^* \) and \( \text{PF}^* \) in GTCO4 are achieved through the implementation of Eqn. 11. Once again, the \( b \) function governed by Eqn. 7 is applied to generate \( \text{PS}^* \) changes and the corresponding Pareto fronts at different \( \sigma \) levels are shown in Fig. 5. At low levels of \( \sigma \), \( \text{PS}^* \) corresponds to \( x_r^* = 0.75 \) and the \( \text{PF}^* \) becomes increasingly nonconvex with noise. At sufficiently high \( \sigma \) levels, the \( \text{PS}^* \) corresponds to \( x_r^* = 0.25 \). Note that nonconvexity is one of the problems that posed considerable difficulties to early MO algorithms. Therefore, a robust MOEA has to be capable of distributing the
discovered solutions uniformly along the Pareto front for the various degrees of convexity. The settings of \( |x_{i2}| = |x_{i1}| = 10 \) and \( \sigma = 0.2 \) are recommended for this problem.

GTCO1 is based on Eqn. 12 which introduces noise-induced PS \(^{A}\) and PF \(^{A}\) changes. Robustness of the solutions are correlated to \( f_1 \) and this presents a conflict with the optimality of \( f_2 \). Considering the effects of this tradeoff alone, the region of PF that remains becomes increasingly smaller with noise as illustrated in Fig. 6(a). Fitness topological changes are based on the principle adopted in GTCO1. However, the associated \( b \) function gives rise to a deceptive landscape in this instance as shown in Fig. 5(b). The only decision variable associated with PS \(^{A}\) that varies with \( \sigma \) is \( x_1 \) while the others remain at \( x_{i2} = 0.0 \) and \( x_{i2} = 0.0 \). The settings of \( |x_{i2}| = |x_{i1}| = 10 \) and \( \sigma = 0.08 \) are recommended for this problem.

IV. EMPIRICAL ANALYSIS

In this section, simulation studies are conducted to analyze the performances of NSGAII on the proposed GTCO1. Thirty independent runs of 500 generations are performed for each of the test problems. Monte Carlo integration based on H samples is applied. Maximum spread (MS) and the variable distance (VD) is used to assess algorithmic performance. VD is given as,

\[
VD = \frac{1}{n_{PS}} \left( \sum_{i=1}^{n_{PS}} d_i^2 \right)^{\frac{1}{2}} \tag{13}
\]

where \( n_{PS} = |PS^{A}| \), \( d_i \) is the Euclidean distance (in decision space) between the \( i \)-th member of PS and the nearest member of PS \(^{A}\).

The performance trend of NSGAII over the number of samples, \( H = \{1, 5, 10, 20 \} \) and \( \sigma = \{0.0, 0.05, 0.1, 0.2 \} \) for GTCO1 is shown in Fig. 7. GTCO1 is a class 1 problem where multimodality is introduced into the fitness landscape with noise while PS\(_{det}^{A} = PS\(_{eff}^{A} \sigma \). From the metric of VD in Fig. 7(a), NSGAII encounter no difficulty finding the PS\(_{det}^{A} \) but faced increasing difficulties with increasing \( \sigma \). The diversity of \( PF\(_{eff}^{A} \sigma \) is also affected. This clearly demonstrates how robust optimization can be more difficult in the face of noise-induced landscape features.

V. CONCLUSION

This paper presents a set of guidelines for the construction of robust MO test problems. The fundamental component of the robust test problems is a Gaussian landscape generator that facilitates the specification of robust optimization-specific features such as noise-induced solution space, fitness landscape and decision space variation. This generator is developed with the purpose of generating noise-sensitive landscapes in conjunction with existing MO test problems, and due to its independent nature, it can be used to generate robust SO test problems as well. Subsequently, a robust continuous MO test suite is built upon the ZDT framework. The study suggests that robust MO problems can offer greater challenges to optimization algorithms when noise is introduced, highlighting the necessity to design more effective MOEAs as well as more rigorous simulation studies.

REFERENCES