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Term Graph Narrowing

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We introduce term graph narrowing as an approach for solving equations by transformations on term graphs. Term graph narrowing combines term graph rewriting with first-order term unification. Our main result is that this mechanism is complete for all term rewriting systems over which term graph rewriting is normalizing and confluent. This includes, in particular, all convergent term rewriting systems. Completeness means that for every solution of a given equation, term graph narrowing can find a more general solution. The general motivation for using term graphs instead of terms is to improve efficiency: sharing common subterms saves space and avoids the repetition of computations.

1. Introduction

Narrowing was devised in the field of theorem proving as an equation solving method for the case when an equational theory is represented by a convergent term rewriting system. Fay (1979) was the first to show the completeness of narrowing. In order to reduce the search space of the narrowing procedure, Hullot (1980) considered a strategy called basic narrowing and showed that it is still complete. Later, narrowing became popular as the computational paradigm for the combination of functional and logic programming. Since then there has been much research activity on improving the efficiency of narrowing and on relaxing the requirements for completeness (see the recent survey of Hanus (1994)).

In order to implement narrowing efficiently, it is advisable to represent terms by graph-like data structures. This is because the simple tree representation of terms enforces copying of subterms in rewrite steps and hence leads to multiplication of evaluation work. In this paper we introduce *term graph narrowing* as an approach for solving equations by transformations on term graphs. Our main result is that term graph narrowing is complete for all term rewriting systems over which term graph rewriting is normalizing and confluent. This includes, in particular, all convergent term rewriting systems. Completeness means that if an equation is represented by a term graph, then for every solution of this equation, term graph narrowing can find a more general solution (that is, every solution is equivalent to an instance of a computed solution).

Term graph narrowing combines term graph rewriting with first-order term unification (see Sleep *et al.* (1993) for a recent collection of papers on term graph rewriting). We use the term graph rewriting model studied in Hoffmann and Plump (1991), Plump (1993a) and Plump (1993b). It allows, besides applications of rewrite rules, collapsing steps on term graphs to increase the degree of sharing. This model is complete with respect to equational deduction in the same sense as term rewriting is. Our completeness proof for term graph narrowing exploits existing results on term graph rewriting for its relationship with term rewriting with respect to termination, confluence and related properties.

Corradini and Wolz (1994) consider narrowing on term graphs with multiple roots, so-called jungles. In their approach, rewriting and unification is based on jungle pushouts, and rewriting includes neither garbage collection nor arbitrary collapsing (jungle pushouts produce a minimal collapsing similar to the special derivations that we consider in Section 4). The results in Corradini and Wolz (1994) aim at showing the correctness of a concrete implementation of (conditional) narrowing, while we are interested in showing completeness of term graph narrowing for equational unification.

This paper is organized as follows: In Section 2, we define term graphs as special hypergraphs and introduce substitutions for term graphs. Term graph rewriting is reviewed in Section 3. In particular, we recall results about soundness, completeness, normalization, and confluence. Section 4 deals with so-called minimally collapsing rewrite derivations which are needed in Section 6 when rewrite derivations are transformed into narrowing derivations. In Section 5, we define term graph narrowing and show its soundness and completeness. The completeness proof is based on the so-called Lifting Lemma which allows to "lift" rewrite derivations to narrowing derivations. The Lifting Lemma itself is established in Section 6. We conclude in Section 7 with a brief summary and with some topics for future work.

2. Term graphs and substitutions

Let Σ be a set of *function symbols*. Each function symbol f comes with a natural number $arity(f) \geq 0$. Function symbols of arity 0 are called *constants*. We further assume that there is an infinite set *Var* of *variables* such that $Var \cap \Sigma = \emptyset$. For each variable x, we set $arity(x) = 0$.

A *hypergraph* over $\Sigma \cup Var$ is a system $G = \langle V_G, E_G, lab_G, att_G \rangle$ consisting of two finite sets V_G and E_G of *nodes* and *hyperedges*, a labelling function $lab_G\colon E_G \to \Sigma \cup Var$, and an attachment function $att_G\colon E_G \to V_G^*$ which assigns a string of nodes to a hyperedge e such that the length of $att_G(e)$ is $1 + arity(lab_G(e))$. In the following we call hypergraphs and hyperedges simply graphs and edges. The set of variables occurring in G is denoted by $Var(G)$, that is, $Var(G) = lab_G(E_G) \cap Var$.

Given a graph G and an edge e with $att_G(e) = v\, v_1 \ldots v_n$, node v is the *result node* of e while $v_1, \ldots, v_n$ are the *argument nodes*. Given two nodes v and v' in G, we write $v >_G^1 v'$ if there is an edge e with result node v such that v' is an argument node of e. The transitive (reflexive-transitive) closure of $>_G^1$ is denoted by $>_G$ ($\geq_G$). G is *acyclic* if $>_G$ is irreflexive. We write $G[v]$ for the graph consisting of all nodes v' with $v \geq_G v'$ and all edges having these nodes as result nodes.

Definition 2.1. (Term graph) A graph G is a *term graph* if

(1) there is a node $root_G$ such that $root_G \geq_G v$ for each node v,
(2) G is acyclic, and
(3) each node is the result node of a unique edge.

Figure 1 shows three term graphs with function symbols `f`, `g`, `a`, and variables `x`, `y`. The symbol `f` is binary, `g` is unary, and `a` is a constant. Edges are depicted as boxes with inscribed labels, and bullets represent nodes. A line connects each edge with its result node, while arrows point to the argument nodes. The order in the argument string is given by the left-to-right order of the arrows leaving the box.

Occasionally we use the following principle of *bottom-up induction* to show that a property P holds for all nodes of a term graph G. This principle is as follows:

(1) show that P holds for all nodes representing constants and variables, then
(2) show that P holds for the result node of an edge if it holds for all argument nodes of this edge.

A *term* over $\Sigma \cup \mathit{Var}$ is a variable, a constant, or a string $f(t_1, \ldots, t_n)$ where f is a function symbol of arity n, $n \geq 1$, and $t_1, \ldots, t_n$ are terms. The *subterms* of a term t are t and, if $t = f(t_1, \ldots, t_n)$, all subterms of $t_1, \ldots, t_n$. We write $\mathit{Var}(t)$ for the set of variables occurring in a term t.

Definition 2.2. (Term representation) A node v in a term graph G represents the term

$$\mathit{term}_G(v) = \mathit{lab}_G(e)(\mathit{term}_G(v_1), \ldots, \mathit{term}_G(v_n)),$$

where e is the unique edge with result node v, and where $\mathit{att}_G(e) = v\, v_1 \ldots v_n$. We write $\mathit{lab}_G(e)$ instead of $\mathit{lab}_G(e)()$ if $v_1 \ldots v_n$ is the empty string.

Note that $\mathit{term}_G(v)$ is well-defined by properties (2) and (3) of Definition 2.1. In the following we abbreviate $\mathit{term}_G(\mathit{root}_G)$ by $\mathit{term}(G)$.

A *graph morphism* $f: G \to H$ between two graphs G and H consists of two functions $f_V: V_G \to V_H$ and $f_E: E_G \to E_H$ that preserve labels and attachment to nodes, that is, $\mathit{lab}_H \circ f_E = \mathit{lab}_G$ and $\mathit{att}_H \circ f_E = f_V^* \circ \mathit{att}_G$ (where $f_V^*: V_G^* \to V_H^*$ maps a string $v_1 \ldots v_n$ to $f_V(v_1) \ldots f_V(v_n)$). We omit the subscripts V and E if no confusion is possible. The morphism f is *injective* (*surjective, bijective*) if f_V and f_E are so. If f is bijective, then it is an *isomorphism*. In this case G and H are *isomorphic*, which is denoted by $G \cong H$. The *composition* $g \circ f$ of two graph morphisms $f: G \to H$ and $g: H \to J$ consists of the composed functions $g_V \circ f_V$ and $g_E \circ f_E$. Given a node v in H, we write $f^{-1}(v)$ for the set $\{\overline{v} \in V_G \mid f(\overline{v}) = v\}$.

Definition 2.3. (Collapsing) Given two term graphs G and H, G *collapses* to H if there is a graph morphism $c: G \to H$ mapping root_G to root_H. This is denoted by $G \succeq_c H$ or simply by $G \succeq H$. We write $G \succ_c H$ or $G \succ H$ if c is non-injective. The latter kind of collapsing is said to be *proper*. A term graph G is *fully collapsed* if there is no H with $G \succ H$.

It is easy to see that the collapse morphisms are the surjective morphisms between term graphs and that $G \succeq H$ implies $\mathit{term}(G) = \mathit{term}(H)$. An example of collapsing is given in Figure 1.

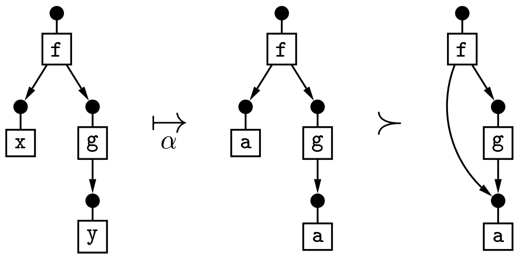


Fig. 1. A substitution application and a collapsing

The term graph substitutions defined next correspond to first-order term substitutions. They are a special case of the general graph substitutions introduced in Plump and Habel (1996), which operate on variable edges with an arbitrary number of attachment nodes.

A *substitution pair* x/G consists of a variable x and a term graph G. Given a term graph H and an edge e in H labelled with x, the application of x/G to e proceeds in two steps:

(1) Remove e from H, yielding the graph $H - \{e\}$, and
(2) construct the disjoint union $(H - \{e\}) + G$ and fuse the result node of e with $root_G$.

It is easy to see that the resulting graph is a term graph.

Definition 2.4. (Term graph substitution) A *term graph substitution* is a finite set $\alpha = \{x_1/G_1, \ldots, x_n/G_n\}$ of substitution pairs such that $x_1, \ldots, x_n$ are pairwise distinct and $x_i \neq \mathit{term}(G_i)$ for $i = 1, \ldots, n$. The *domain* of α is the set $\mathit{Dom}(\alpha) = \{x_1, \ldots, x_n\}$. The application of α to a term graph H yields the term graph $H\alpha$ which is obtained by applying all substitution pairs in α simultaneously to all edges with label in $\mathit{Dom}(\alpha)$.

Example 2.5. Let $\mathtt{x}, \mathtt{y}$ be variables and A be a term graph with $\mathit{term}(A) = \mathtt{a}$. An application of the substitution $\alpha = \{\mathtt{x}/A,\ \mathtt{y}/A\}$ is shown in Figure 1.

Given a term graph G, we write $\underline{G}$ for the graph that results from removing all edges labelled with variables. If α is a term graph substitution, we assume for technical convenience that the unique graph morphism $in\colon \underline{G} \to G\alpha$ with $in(root_G) = root_{G\alpha}$ satifies $in(a) = a$ for all nodes and edges a.

Lemma 2.6. Let G be a term graph and α a term graph substitution. Then for each node v in G, $G\alpha[v] \cong G[v]\alpha$.

Proof. Let the "garbage" of G consist of all nodes and edges that are not reachable from v. The proposition follows from the fact that the application of a substitution pair to a variable edge that is not in the garbage can be commuted with the removal of the garbage. □

Definition 2.7. (Composition of term graph substitutions) Given two term graph substitutions α and β, the *composition* $\alpha\beta$ is defined by

$$\alpha\beta = \{x/G\beta \mid x/G \in \alpha \text{ and } x \neq \mathit{term}(G\beta)\} \cup \{y/H \in \beta \mid y \notin \mathit{Dom}(\alpha)\}.$$

The following property of the composition is an immediate consequence of the associativity of edge replacement (see Habel (1992)).

Lemma 2.8. For all term graphs G and term graph substitutions α and β,

$$G(\alpha\beta) \cong (G\alpha)\beta.$$

Recall that a *term substitution* σ is a finite set $\sigma = \{x_1/t_1, \ldots, x_n/t_n\}$, where $x_1, \ldots, x_n$ are pairwise distinct variables and $t_1, \ldots, t_n$ are terms such that $x_i \neq t_i$ for $i = 1, \ldots, n$. The application of σ to a term t yields the term $t\sigma$ which is obtained from t by simultaneously replacing each occurrence of the variable x_i by t_i ($i = 1, \ldots, n$). The composition of term substitutions is defined analogously to the composition of term graph substitutions.

Two terms s and t are *unifiable* if there exists a term substitution σ such that $s\sigma = t\sigma$. In this case σ is called a *unifier* of s and t. Each two unifiable terms s and t possess a *most general* unifier σ, having the property that for every other unifier τ of s and t there is a term substitution ρ such that $\sigma\rho = \tau$. Moreover, there are unification algorithms ensuring that σ is *idempotent*, meaning $\sigma\sigma = \sigma$ (see, for example, Apt (1990)).

Definition 2.9. (Induced term substitution) For every term graph substitution α, the *induced* term substitution α^{term} is defined by

$$\alpha^{term} = \{x/term(G) \mid x/G \in \alpha\}.$$

Lemma 2.10. Let G be a term graph and α be a term graph substitution. Then for each node v in G,

$$term_{G\alpha}(v) = term_G(v)\alpha^{term}.$$

Proof. By induction on the structure of G. The proposition is obvious if v represents a constant or a variable that is not in $Dom(\alpha)$. If $term_G(v)$ is a variable x such that $x/H \in \alpha$, then $term_{G\alpha}(v) = term(H) = x\{x/term(H)\} = term_G(v)\alpha^{term}$. Now consider the unique edge e with result node v. Let $att_G(e) = v\,v_1 \ldots v_n$ with $n \geq 1$ and suppose that v_i satisfies the proposition, for $i = 1, \ldots, n$. Then

$$\begin{aligned}
term_{G\alpha}(v) &= lab_{G\alpha}(e)(term_{G\alpha}(v_1), \ldots, term_{G\alpha}(v_n)) \\
&= lab_G(e)(term_{G\alpha}(v_1), \ldots, term_{G\alpha}(v_n)) \\
&= lab_G(e)(term_G(v_1)\alpha^{term}, \ldots, term_G(v_n)\alpha^{term}) \\
&= lab_G(e)(term_G(v_1), \ldots, term_G(v_n))\alpha^{term} \\
&= term_G(v)\alpha^{term}.
\end{aligned}$$

□

Lemma 2.11. For all term graph substitutions α and β,

$$(\alpha\beta)^{term} = \alpha^{term}\beta^{term}.$$

Proof. The proposition follows from Lemmas 2.10 and 2.8. □

3. Term graph rewriting

In this section we review the term graph rewriting model investigated in Hoffmann and Plump (1991), Plump (1993a) and Plump (1993b). In particular, we state the results that are needed for proving the completeness of term graph narrowing in the next section.

These include the soundness and completeness of term graph rewriting and the relation to term rewriting with respect to normalization and confluence.

We first recall some properties of relations and basic concepts of term rewriting systems (more information can be found, for example, in the surveys of Dershowitz and Jouannaud (1990) and Klop (1992)). Let A be a set and $\to$ be a binary relation on A. Then $\to^*$ and $\leftrightarrow^*$ denote the transitive-reflexive and symmetric-transitive-reflexive closures of $\to$. The relation $\leftrightarrow^*$ is called *conversion*, and two elements a and b with $a \leftrightarrow^* b$ are *convertible*. The inverse relation of $\to$ is denoted by $\leftarrow$. The relation $\to$ is *confluent* if for all a, b, c with $b \leftarrow^* a \to^* c$ there is some d such that $b \to^* d \leftarrow^* c$. The relation $\to$ is *terminating* if there is no infinite sequence $a_1 \to a_2 \to a_3 \to \ldots$ If $\to$ is confluent and terminating, then it is *convergent*. An element a in A is a *normal form* if there is no b such that $a \to b$. Element a *has a normal form* if $a \to^* b$ for some normal form b. The relation $\to$ is *normalizing* if each element in A has a normal form.

A *term rewrite rule* $l \to r$ consists of two terms l and r such that

(1) l is not a variable and
(2) all variables in r occur also in l.

Such a rule is *left-linear* if no variable occurs more than once in l. A set $\mathcal{R}$ of term rewrite rules is a *term rewriting system*. The term rewrite relation $\to_{\mathcal{R}}$ associated with $\mathcal{R}$ is defined as follows: $t \to_{\mathcal{R}} u$ if there is a rule $l \to r$ in $\mathcal{R}$ and a term substitution σ such that

(1) $l\sigma$ is a subterm of t and
(2) u is obtained from t by replacing an occurrence of $l\sigma$ by $r\sigma$.

Let v be a node in a term graph G. Define $\mathit{indegree}_G(v) = \sum_{e \in E_G} \#_v(e)$, where for each edge e with $\mathit{att}_G(e) = v_0 v_1 \ldots v_n$, $\#_v(e)$ is the number of occurrences of v in $v_1 \ldots v_n$.

Definition 3.1. For every term t, let $\Diamond t$ be a term graph representing t such that only variable nodes are shared, that is,

(1) $\mathit{indegree}_G(v) \leq 1$ for each node v with $\mathit{term}_{\Diamond t}(v) \notin \mathit{Var}$, and
(2) $v_1 = v_2$ for all nodes v_1,v_2 with $\mathit{term}_{\Diamond t}(v_1) = \mathit{term}_{\Diamond t}(v_2) \in \mathit{Var}$.

Recall, for the next lemma, that $\underline{\Diamond t}$ is the graph obtained from $\Diamond t$ by removing all variable edges. The lemma shows that a term rewrite rule $l \to r$ can be applied to the term represented by a term graph G whenever there is a graph morphism $\underline{\Diamond l} \to G$.

Lemma 3.2. Let l be a term and G be a term graph. Then for each graph morphism $f\colon \underline{\Diamond l} \to G$ there is a term substitution σ such that $\mathit{term}_G(f(\mathit{root}_{\Diamond l})) = l\sigma$.

Proof. Let $\sigma = \{x/\mathit{term}_G(f(v_x)) \mid x \in \mathit{Var}(l) \text{ and } x \neq \mathit{term}_G(f(v_x))\}$, where v_x is the unique node with $\mathit{term}_{\Diamond l}(v_x) = x$. Then a straightforward bottom-up induction on G yields the proposition. □

Definition 3.3. (Redex and preredex) Let G be a term graph, v be a node in G, and $l \to r$ be a rule in $\mathcal{R}$. The pair $\langle v, l \to r\rangle$ is a *redex* if there is a graph morphism $\mathit{red}\colon \underline{\Diamond l} \to G$, called the *redex morphism*, such that $\mathit{red}(\mathit{root}_{\Diamond l}) = v$. The pair $\langle v, l \to r\rangle$ is a *preredex* if there is a term substitution σ such that $\mathit{term}_G(v) = l\sigma$.

By Lemma 3.2, every redex is a preredex. The converse also holds if the rule $l \to r$ is

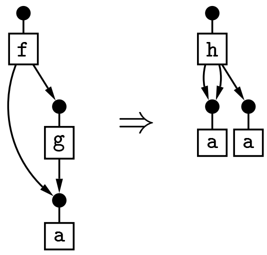


Fig. 2. A term graph rewrite step

left-linear, since then $\Diamond l$ is a tree[1]. However, if l contains repeated variables, then there need not exist a graph morphism sending $root_{\Diamond l}$ to v. In this case a suitable collapsing of G turns the preredex $\langle v,\, l \to r\rangle$ into a redex.

Definition 3.4. (Term graph rewriting) Let G, H be term graphs and $\langle v,\, l \to r\rangle$ be a redex in G with redex morphism $red\colon \underline{\Diamond l} \to G$. Then there is a *proper rewrite step* $G \Rightarrow_{v,\, l\to r} H$ if H is isomorphic to the term graph G_3 constructed as follows:

(1) $G_1 = G - \{e\}$ is the graph obtained from G by removing the unique edge e having result node v.

(2) G_2 is the graph obtained from the disjoint union $G_1 + \underline{\Diamond r}$ by
  — identifying v with $root_{\Diamond r}$,
  — identifying $red(v_1)$ with v_2, for each pair $\langle v_1, v_2\rangle \in V_{\Diamond l} \times V_{\Diamond r}$ with $term_{\Diamond l}(v_1) = term_{\Diamond r}(v_2) \in Var$.

(3) $G_3 = G_2[root_G]$ is the term graph obtained from G_2 by removing all nodes and edges not reachable from $root_G$ ("garbage collection").

We define the term graph rewrite relation $\Rightarrow_{\mathcal{R}}$ by adding proper collapse steps: $G \Rightarrow_{\mathcal{R}} H$ if $G \succ H$ or $G \Rightarrow_{v,\, l\to r} H$ for some redex $\langle v,\, l \to r\rangle$.

A *term graph rewrite derivation* is either an isomorphism $G \to H$, which is a derivation of length 0, or a non-empty sequence

$$G = G_0 \Rightarrow_{\mathcal{R}} G_1 \Rightarrow_{\mathcal{R}} \ldots \Rightarrow_{\mathcal{R}} G_n = H.$$

We denote such a derivation (ambiguously) by $G \Rightarrow^*_{\mathcal{R}} H$.[2]

Example 3.5. A term graph rewrite step with rule $\mathtt{f(x, g(x))} \to \mathtt{h(x, x, a)}$ is given in Figure 2. Note that the left term graph is obtained from the middle term graph of Figure 1 by collapsing. However, the rule is not applicable to the middle term graph of Figure 1 because there is no graph morphism from $\underline{\Diamond \mathtt{f(x, g(x))}}$ into that graph.

Next we define a function which allows to "follow" the nodes of an initial term graph G through a derivation $G \Rightarrow^*_{\mathcal{R}} H$.

[1] A term graph is a *tree* if all nodes but the root have indegree one.
[2] By considering $\Rightarrow_{\mathcal{R}}$ as a relation on isomorphism classes of term graphs, this notation does not conflict with the definition of $\Rightarrow^*_{\mathcal{R}}$ as the transitive-reflexive closure of $\Rightarrow_{\mathcal{R}}$.

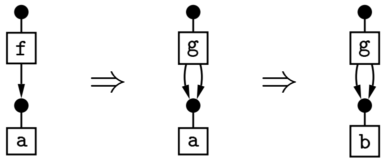


Fig. 3. A term graph rewrite derivation

Definition 3.6. (Track function) Let $G \Rightarrow_{v,l\to r} H$ be a proper term graph rewrite step. Let, in the construction of Definition 3.4, $in: G_1 \to G_1 + \underline{\Diamond r}$ be the injective graph morphism associated with the disjoint union and $ident: G_1 + \underline{\Diamond r} \to G_2$ be the surjective morphism associated with the identification. Moreover, let $iso: G_3 \to H$ be the isomorphism between G_3 and H. Then the *track function* associated with this rewrite step is the partial function $tr_{G\Rightarrow H}: V_G \to V_H$ defined as follows:[3]

$$tr_{G\Rightarrow H}(v) = \begin{cases} iso(ident(in(v))) & \text{if } ident(in(v)) \in V_{G_3}, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

If $G \Rightarrow_{\mathcal{R}} H$ is a collapse step $G \succ_c H$, then $tr_{G\Rightarrow H} = c_V$. We extend the track function to derivations as follows: If $G \Rightarrow^*_{\mathcal{R}} H$ is an isomorphism $i: G \to H$, then $tr_{G\Rightarrow^* H} = i_V$. If $G \Rightarrow^*_{\mathcal{R}} H$ has the form $G \Rightarrow_{\mathcal{R}} M \Rightarrow^*_{\mathcal{R}} H$, then $tr_{G\Rightarrow^* H} = tr_{M\Rightarrow^* H} \circ tr_{G\Rightarrow M}$.

The term graph rewrite relation $\Rightarrow_{\mathcal{R}}$ is sound with respect to term rewriting in the sense that every proper step $G \Rightarrow_{v,l\to r} H$ corresponds to a parallel application of $l \to r$ to several occurrences of the subterm $term_G(v)$ in $term(G)$. This parallelism is the reason for the possible speed-up of term graph rewriting with respect to term rewriting. Note that if $G \Rightarrow_{\mathcal{R}} H$ is a collapse step, then $term(G) = term(H)$ and hence $term(G) \to^*_{\mathcal{R}} term(H)$.

Theorem 3.7. (Soundness of rewriting (Hoffmann and Plump 1991)) For all term graphs G and H,

$$G \underset{\mathcal{R}}{\Rightarrow} H \text{ implies } term(G) \overset{*}{\underset{\mathcal{R}}{\to}} term(H).$$

The converse of the implication (with $\Rightarrow_{\mathcal{R}}$ replaced by $\Rightarrow^*_{\mathcal{R}}$) does not hold since certain term rewrite derivations do not correspond to term graph rewrite derivations. Consider, for example, the rules $\mathtt{f(x)} \to \mathtt{g(x,x)}$ and $\mathtt{a} \to \mathtt{b}$. The derivation $\mathtt{f(a)} \to_{\mathcal{R}} \mathtt{g(a,a)} \to_{\mathcal{R}} \mathtt{g(a,b)}$ cannot be simulated by term graph rewriting since the application of the first rule leads to a shared constant $\mathtt{a}$. Hence, as shown in Figure 3, applying the second rule results in a term graph representing $\mathtt{g(b,b)}$.

Although not all term derivations possess corresponding graph derivations, the conversion of term graphs is complete with respect to conversion of terms.

Theorem 3.8. (Completeness of rewriting (Plump 1993a)) For all term graphs G and H,

$$term(G) \overset{*}{\underset{\mathcal{R}}{\leftrightarrow}} term(H) \text{ if and only if } G \overset{*}{\underset{\mathcal{R}}{\Leftrightarrow}} H.$$

[3] Note that $tr_{G\Rightarrow H}$ is well-defined since $V_G \subseteq V_{G_1}$ and $V_{G_3} \subseteq V_{G_2}$.

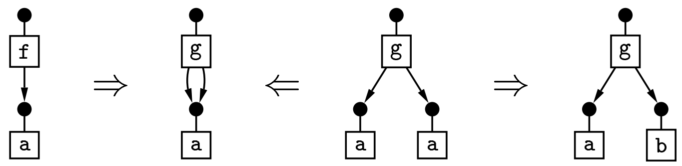


Fig. 4. A term graph conversion

For instance, in Figure 4 the term graph representing `f(a)` is converted into the term graph representing `g(a, b)`. Note that the rewrite step in the middle is a collapse step. Indeed, a conversion between the outer term graphs without collapse steps is impossible. That is, the above completeness result does not hold for proper rewrite steps.

The next two theorems explain the relationship between term graph rewriting and term rewriting with respect to normalization, confluence, and convergence. These results are used in proving the completeness of term graph narrowing.

Theorem 3.9. (Normalization (Hoffmann and Plump 1991))

1. A term graph G is a normal form with respect to $\Rightarrow_\mathcal{R}$ if and only if G is fully collapsed and $term(G)$ is a normal form with respect to $\rightarrow_\mathcal{R}$.
2. If $\Rightarrow_\mathcal{R}$ is normalizing, then so is $\rightarrow_\mathcal{R}$.

The converse of the second statement does not hold (see Plump (1993b) for a counterexample).

Theorem 3.10. (Confluence and convergence (Plump 1993a))

1. If $\Rightarrow_\mathcal{R}$ is confluent, then so is $\rightarrow_\mathcal{R}$.
2. If $\rightarrow_\mathcal{R}$ is convergent, then so is $\Rightarrow_\mathcal{R}$.

For both statements, the converse does not hold. A counterexample to the converse of the second statement is provided by the following system:

$$\mathcal{R}\left\{\begin{array}{rcl} \mathtt{f(x)} & \rightarrow & \mathtt{g(x,x)} \\ \mathtt{a} & \rightarrow & \mathtt{b} \\ \mathtt{g(a,b)} & \rightarrow & \mathtt{f(a)} \end{array}\right.$$

Here $\Rightarrow_\mathcal{R}$ can be shown to be terminating and confluent, but $\rightarrow_\mathcal{R}$ is non-terminating due to the infinite sequence $\mathtt{f(a)} \rightarrow_\mathcal{R} \mathtt{g(a,a)} \rightarrow_\mathcal{R} \mathtt{g(a,b)} \rightarrow_\mathcal{R} \mathtt{f(a)} \rightarrow_\mathcal{R} \ldots$

We conclude this section with two lemmas about the interaction of term graph rewriting and substitutions.

Lemma 3.11. (Extension Lemma) For all term graphs G, H and term graph substitutions α, $G \Rightarrow_\mathcal{R}^* H$ implies $G\alpha \Rightarrow_\mathcal{R}^* H\alpha$.

Proof. The proposition follows from the Extension Lemma in Plump (1993b). □

Lemma 3.12. (Restriction Lemma) Let $G \Rightarrow_{v,l\to r} H$ be a proper rewrite step with redex morphism $red\colon \underline{\Diamond l} \to G$ and G' be a term graph such that $G'\alpha = G$ for some term graph substitution α. If $red(\underline{\Diamond l}) \subseteq \underline{G'}$, then there is a proper rewrite step $G' \Rightarrow_{v,l\to r} H'$ such that $H'\alpha = H$.

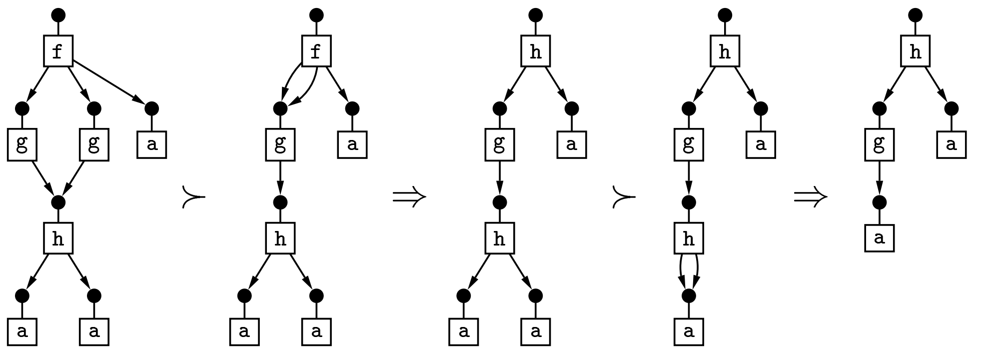


Fig. 5. A minimally collapsing rewrite derivation

Proof. The proposition follows from the Restriction Lemma in Plump (1993b). □

4. Minimally Collapsing Rewrite Derivations

In this section we show that derivability with respect to the rewrite relation $\Rightarrow_{\mathcal{R}}$ is not affected if one restricts to "minimally collapsing" derivations with a subsequent collapsing. In a minimally collapsing derivation, collapse steps are only used to turn preredexes of non-left-linear rewrite rules into redexes. This result is exploited in Section 6 to prove the so-called Lifting Lemma.

Definition 4.1. (Minimal collapsing) A collapsing $G \succeq M$ is *minimal* with respect to a redex $\langle v, l \to r\rangle$ in M if for each term graph M' with $G \succeq M' \succ M$ and each preimage v' of v in M', the preredex $\langle v', l \to r\rangle$ is not a redex.

In particular, if G and M are isomorphic, then $G \succeq M$ is minimal since no M' with $G \succeq M' \succ M$ exists. A proper collapsing $G \succ M$ is minimal only if $l \to r$ is non-left-linear and cannot be applied at any preimage of v in G.

Definition 4.2. (Minimally collapsing rewrite derivation) A rewrite derivation $P \Rightarrow^*_{\mathcal{R}} Q$ is *minimally collapsing* if each collapse step $G \succ M$ in the derivation is followed by a proper rewrite step $M \Rightarrow_{v,\, l \to r} N$ such that $G \succ M$ is minimal with respect to $\langle v, l \to r\rangle$.

Example 4.3. Figure 5 shows a minimally collapsing rewrite derivation with two proper rewrite steps. The applied rules are $\mathtt{f}(\mathtt{x},\mathtt{x},\mathtt{y}) \to \mathtt{h}(\mathtt{x},\mathtt{y})$ and $\mathtt{h}(\mathtt{x},\mathtt{x}) \to \mathtt{x}$.

In order to transform arbitrary rewrite derivations into minimally collapsing derivations, we factorize collapsings into a minimal collapsing and a remaining collapsing.

Lemma 4.4. (Collapse factorization) Let $G \succeq_c H$ be a collapsing and $\langle v, l \to r\rangle$ be a redex in H. Then there exist collapsings $G \succeq_d M \succeq_e H$ and a preimage $\overline{v}$ of v in M such that $\langle \overline{v}, l \to r\rangle$ is a redex and $G \succeq_d M$ is minimal with respect to this redex.

$$\begin{tikzcd}
\underline{\Delta l} \arrow[r, "g"] \arrow[d, "f"'] & \underline{\Diamond l} \arrow[d, dashed, "red'"'] \arrow[dr, "red"] & \\
G \arrow[r, dashed, "d"] \arrow[drr, "c"'] & M \arrow[dr, dashed, "e"] & \\
& & H
\end{tikzcd}$$

Fig. 6. Construction of a minimal collapsing

Proof. If there exists a redex $\langle \overline{v},\, l \to r\rangle$ in G with $c(\overline{v}) = v$, then we simply choose $M = G$, $e = c$, and let d be the identity on G. Assume therefore that no preredex $\langle v',\, l \to r\rangle$ with $c(v') = v$ is a redex.
Let v' be some node in G with $c(v') = v$. Then $term_G(v') = term_H(v)$. Let $red\colon \underline{\Diamond l} \to H$ be the redex morphism for $\langle v,\, l \to r\rangle$. By Lemma 3.2, $term_G(v') = term_H(v) = l\sigma$ for some substitution σ. Hence there is a unique graph morphism $f\colon \underline{\Delta l} \to G$ with $f(root_{\Delta l}) = v'$, where Δl is a tree representing l. Moreover, we have $c \circ f = red \circ g$ for the unique morphism $g\colon \underline{\Delta l} \to \underline{\Diamond l}$ (see Figure 6).
Now a collapsing $G \succ_d M$ is constructed in two steps. At first, a graph G' is obtained from G by identifying $f(v_1)$ with $f(v_2)$ for each two nodes v_1, v_2 in $\underline{\Delta l}$ satisfying $term_{\Delta l}(v_1) = term_{\Delta l}(v_2) \in Var$ (note that $f(v_1)$ and $f(v_2)$ represent the same term). Then the term graph M is obtained from G' by applying the following operation as long as possible: given two distinct edges e_1 and e_2 with the same result node, identify e_1 with e_2 (thereby identifying the i-th argument nodes of e_1 and e_2, for $i = 1, \ldots, arity(lab_{G'}(e_1))$). This operation is well-defined since e_1 and e_2 have the same label. The composition of the two surjective graph morphisms $G \to G' \to M$ associated with the identifications is the collapse morphism d.
The redex morphism $red'\colon \underline{\Diamond l} \to M$ is defined by $red'(a) = d(f(\overline{a}))$, where $\overline{a}$ is an item in $\underline{\Delta l}$ with $g(\overline{a}) = a$. This morphism is well-defined since, by construction of d, $d \circ f$ identifies each two nodes that are identified by g. Moreover, $d \circ f = red' \circ g$ holds by construction. Let $\overline{v} = d(v')$. It is not hard to see that $G \succ_d M$ is minimal with respect to the redex $\langle \overline{v},\, l \to r\rangle$: the identification $G \to G'$ is the minimal one to turn $\langle v',\, l \to r\rangle$ into a redex, and the identification $G' \to M$ is necessary to transform G' into a term graph.
Finally, the collapsing $e\colon M \to H$ sends each item m in M to $c(\overline{m})$, where $\overline{m}$ is some item in G with $d(\overline{m}) = m$. To see that e is well-defined, let w_1', w_2' be two distinct nodes in G with $d(w_1') = d(w_2')$. We have to show $c(w_1') = c(w_2')$. By construction of d, there must be nodes w_1, w_2 in G with preimages v_1, v_2 in $\underline{\Delta l}$ such that $w_i \geq_G w_i'$ for $i = 1, 2$ and $term_{\Delta l}(v_1) = term_{\Delta l}(v_2) \in Var$. Then $g(v_1) = g(v_2)$ and hence $c(w_1) = c(f(v_1)) = red(g(v_1)) = red(g(v_2)) = c(f(v_2)) = c(w_2)$. This implies $c(w_1') = c(w_2')$ since w_1' and w_2' represent the same term and are reachable from w_1 and w_2, respectively.
The graph morphisms of this proof yield the commutative diagram in Figure 6. □

Given a collapsing and a subsequent rewrite step, the collapsing can be "shifted" behind the rewrite step provided that the redex node has only one preimage.

Lemma 4.5. (Collapse shifting) Let $G \succeq_c G' \Rightarrow_{v,\, l\to r} H$ be a collapsing followed by a proper rewrite step such that $c^{-1}(v)$ contains a single node $\overline{v}$. If $\langle \overline{v},\, l \to r\rangle$ is a redex, then there is a term graph H' such that $G \Rightarrow_{\overline{v},\, l\to r} H' \succeq H$.

Proof. Consider the rewrite step $G \Rightarrow_{\overline{v},\, l\to r} H'$ (note that G, $\overline{v}$ and $l \to r$ determine H' uniquely up to isomorphism). To define a collapsing $H' \succeq H$, we exploit that every track function $tr_{A\Rightarrow B}$ can be extended to a partial graph morphism from A to B, that is, to a graph morphism from a subgraph of A to B (just repeat Definition 3.6 for edges). Define the collapse morphism $d\colon H' \to H$ as follows:

$$d(a) = \begin{cases} tr_{G'\Rightarrow H}(c(\overline{a})) & \text{if there is } \overline{a} \text{ with } tr_{G\Rightarrow H'}(\overline{a}) = a, \\ a & \text{otherwise}^4. \end{cases}$$

This morphism is well-defined by two facts: Firstly, $tr_{G\Rightarrow H'}$ identifies two distinct items only if one of the two is the redex node $\overline{v}$ and the other, say w, is the image of a variable node[5] from $\underline{\Diamond l}$. In this case $tr_{G'\Rightarrow H}$ identifies the redex $c(\overline{v})$ with $c(w)$. Secondly, if $tr_{G\Rightarrow H'}(\overline{a})$ is defined, then either there is a path from $root_G$ to $\overline{a}$ not going through $\overline{v}$ (except if $\overline{a} = \overline{v}$), or there is a path from the image w of a variable node in $\underline{\Diamond l}$ to $\overline{a}$ where $tr_{G\Rightarrow H'}(w)$ is defined. Since paths of the former kind are preserved by c and $tr_{G'\Rightarrow H}$, and since $tr_{G'\Rightarrow H}$ is defined for a node like w (there is a path from $tr_{G'\Rightarrow H}(v)$ to the image of w), it follows that $tr_{G'\Rightarrow H}(c(\overline{a}))$ is also defined. □

An example for collapse shifting is shown in Figure 7, where the collapsing on the left and the rewrite step on the bottom are given. The applied rewrite rule is $\mathtt{h(x,x)} \to \mathtt{x}$.

If a collapsing identifies two or more preimages of the redex node of a subsequent rewrite step, the rewrite step has to be "duplicated" in order to obtain a minimally collapsing derivation.

Lemma 4.6. (Duplication of rewrite steps) Let $G \succeq_c G' \Rightarrow_{v,\, l\to r} H$ be a collapsing followed by a proper rewrite step such that $c^{-1}(v)$ contains at least two nodes. Then for each node $\overline{v}$ in $c^{-1}(v)$ there are term graphs G_1, G_2 and $H^{\#}$ such that

$$G \underset{d}{\succeq} G_1 \underset{v_1,\, l\to r}{\Rightarrow} G_2 \underset{v_2',\, l\to r}{\Rightarrow} H^{\#} \succeq H,$$

where $d^{-1}(v_1) = \{\overline{v}\}$.

Proof. Let $c^{-1}(v) = \{\overline{v}, u_1, \ldots, u_n\}$ with $n \geq 1$. We first factorize $G \succeq_c G'$ into collapsings $G \succeq_d G_1 \succeq_{d'} G'$ such that $d(\overline{v}) \neq d(u_i)$ for $i = 1, \ldots, n$ and $d(u_i) = d(u_j)$ for all $i, j \in \{1, \ldots, n\}$. To this end, let S be the subgraph of G consisting of all nodes s with $s \not\geq_G \overline{v}$ and all edges having these nodes as result nodes. Note that S contains $u_1, \ldots, u_n$ since all nodes in $c^{-1}(v)$ represent the same term. Now construct G_1 from G by identifying each two items s_1 and s_2 in S satisfying $c(s_1) = c(s_2)$. Let $d\colon G \to G_1$

[4] For simplicity we assume that each item of $\underline{\Diamond r}$ (apart from $root_{\Diamond r}$ and nodes representing variables in $\Diamond r$) has the same name in H' and H.

[5] A *variable node* is a node representing a variable.

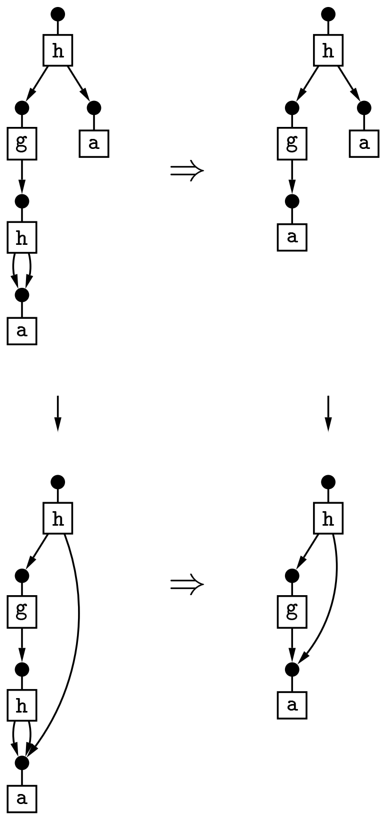


Fig. 7. Shifting a collapse step

be the graph morphism associated with the identification. Define $d'\colon G_1 \to G'$ by the assignment $a \mapsto c(\overline{a})$, where $\overline{a}$ is an item in G with $d(\overline{a}) = a$.
Let $d(\overline{v}) = v_1$ and $d(u_i) = v_2$ for $i = 1, \ldots, n$. By definition of d, we have $d^{-1}(v_1) = \{\overline{v}\}$. By construction of G_1, $\langle v_1, l \to r\rangle$ and $\langle v_2, l \to r\rangle$ are redexes. Moreover, we have $v_1 \not\geq_{G_1} v_2$ and $v_2 \not\geq_{G_1} v_1$. Hence there are rewrite steps $G_1 \Rightarrow_{v_1, l\to r} G_2 \Rightarrow_{v_2', l \to r} H^\#$ with $v_2' = tr_{G_1 \Rightarrow G_2}(v_2)$. To define the collapsing $H^\# \succeq H$, we consider $tr_{G_1 \Rightarrow G_2 \Rightarrow H^\#}$ as a partial morphism from G_1 to $H^\#$ (see also the proof of the preceding lemma). By the construction of rewrite steps, each item in $H^\#$ that is not in $tr_{G_1 \Rightarrow G_2 \Rightarrow H^\#}(G_1)$ stems from one of the two insertions of $\underline{\Diamond r}$ in $G_1 \Rightarrow G_2 \Rightarrow H^\#$. For each such item a, let $h(a)$ be the corresponding item in H inserted by the step $G' \Rightarrow H$. Then the collapse morphism $f\colon H^\# \to H$ is defined by

$$f(a) = \begin{cases} tr_{G' \Rightarrow H}(d'(\overline{a})) & \text{if there is } \overline{a} \text{ with } tr_{G_1 \Rightarrow G_2 \Rightarrow H^\#}(\overline{a}) = a, \\ h(a) & \text{otherwise.} \end{cases}$$

That f is well-defined follows by reasons similar to those given in the proof of the preceding lemma. $\square$

$$
\begin{tikzcd}
G \arrow[rr, Rightarrow, "\overline{v}, l \to r"'] \arrow[dr, "d"] \arrow[dd, two heads, "c"'] \arrow[drr, phantom, "\text{Lemma 4.5}"] & & G^{\#} \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\text{Lemma 4.4}"] & M \arrow[r, Rightarrow, "\overline{v}_2, l \to r"'] \arrow[dl] \arrow[drr, phantom, "\text{induction hypothesis}"] & M^{\#} \arrow[r, Rightarrow, "*", "\mathcal{R}"'] & H' \arrow[d] \\
& G_1 \arrow[r, Rightarrow, "v_1, l \to r"'] \arrow[dl] \arrow[drr, phantom, "\text{Lemma 4.6}"] & G_2 \arrow[rrr, Rightarrow, "v'_2, l \to r"'] & & & H^{\#} \arrow[d] \\
G' \arrow[rrrrr, Rightarrow, "v, l \to r"'] & & & & & H
\end{tikzcd}
$$

Fig. 8. Structure of the proof of Lemma 4.7

Now we can transform pairs of collapse and rewrite steps into minimally collapsing derivations with subsequent collapsings.

Lemma 4.7. (Transformation of rewrite steps) Let $G \succeq G' \Rightarrow_{v,\,l\to r} H$ be a collapsing followed by a proper rewrite step, and $\overline{v}$ be a preimage of v in G such that $\langle \overline{v},\, l \to r\rangle$ is a redex. Then there is a minimally collapsing derivation $G \Rightarrow_{\overline{v},\,l\to r} G^{\#} \Rightarrow^*_{\mathcal{R}} H'$ such that $H' \succeq H$.

Proof. Let $G \succeq_c G'$. We proceed by induction on the number of nodes in $c^{-1}(v)$.
Induction base. If $\overline{v}$ is the only node in $c^{-1}(v)$, then the proposition holds by Lemma 4.5 (choose $G^{\#} = H'$).
Induction step. Let $c^{-1}(v)$ contain $n+1$ nodes, with $n \geq 1$. By Lemma 4.6, there are term graphs G_1, G_2 and $H^{\#}$ such that $G \succeq_d G_1 \Rightarrow_{v_1,\,l\to r} G_2 \Rightarrow_{v'_2,\,l\to r} H^{\#} \succeq H$, where $d^{-1}(v_1) = \{\overline{v}\}$. Hence, by Lemma 4.5, there is a term graph $G^{\#}$ such that

$$G \underset{\overline{v},\,l\to r}{\Rightarrow} G^{\#} \underset{g}{\succeq} G_2.$$

From the proof of Lemma 4.5 we know that $g^{-1}(v'_2) = tr_{G\Rightarrow G^{\#}}(d^{-1}(v_2))$, where v_2 is the unique node with $tr_{G_1\Rightarrow G_2}(v_2) = v'_2$. By the proof of Lemma 4.6, $d^{-1}(v_2)$ contains n nodes. Hence $g^{-1}(v'_2)$ contains also n nodes ($tr_{G\Rightarrow G^{\#}}$ is injective on all nodes different from $\overline{v}$ and does not identify $\overline{v}$ with any node in $d^{-1}(v_2)$). Now we factorize $G^{\#} \succeq_g G_2$ according to Lemma 4.4. This yields collapsings $G^{\#} \succeq M \succeq G_2$ with M containing a redex $\langle \overline{v}_2,\, l \to r\rangle$ such that $\overline{v}_2$ is a preimage of v'_2 and such that $G^{\#} \succeq M$ is minimal with respect to the redex. It follows that v'_2 has n preimages in M. Thus, by induction hypothesis, there is a minimally collapsing derivation $M \Rightarrow_{\overline{v}_2,\,l\to r} M^{\#} \Rightarrow^*_{\mathcal{R}} H'$ such that $H' \succeq H^{\#}$. To conclude, we have obtained a minimally collapsing derivation

$$G \underset{\overline{v},\,l\to r}{\Rightarrow} G^{\#} \succeq M \underset{\overline{v}_2,\,l\to r}{\Rightarrow} M^{\#} \Rightarrow^*_{\mathcal{R}} H'$$

such that $H' \succeq H^{\#} \succeq H$. This completes the proof. □

Figure 8 illustrates the structure of the proof of Lemma 4.7. Using this lemma, it is straightforward to establish the desired transformation theorem for arbitrary derivations.

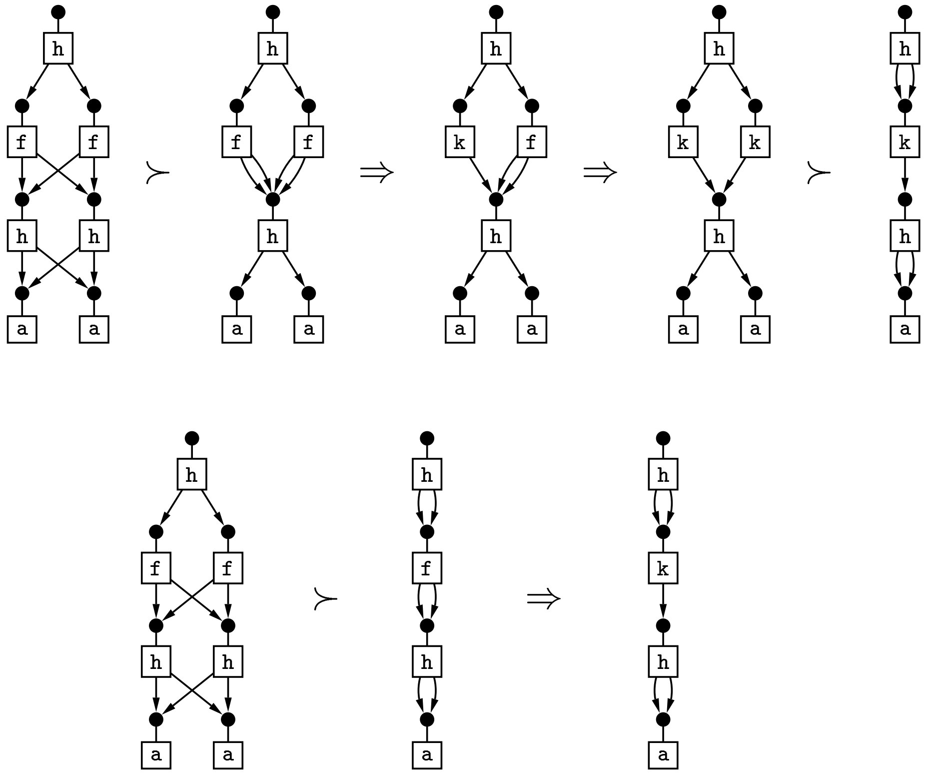


Fig. 9. Transformation of a rewrite derivation

Theorem 4.8. (Transformation of derivations) For every derivation $G \Rightarrow_{\mathcal{R}}^* H$ there is a minimally collapsing derivation $G \Rightarrow_{\mathcal{R}}^* H'$ such that $H' \succeq H$.

Proof. The proposition holds trivially if $G \Rightarrow_{\mathcal{R}}^* H$ is a derivation of length 0. Otherwise there is a term graph J such that $G \Rightarrow_{\mathcal{R}}^* J \Rightarrow_{\mathcal{R}} H$. We may assume, as induction hypothesis, that there is a minimally collapsing derivation $G \Rightarrow_{\mathcal{R}}^* J'$ such that $J' \succeq J$. If $J \Rightarrow_{\mathcal{R}} H$ is a collapse step, then $G \Rightarrow_{\mathcal{R}}^* J' \succeq H$ and we are done. Assume therefore that $J \Rightarrow_{\mathcal{R}} H$ is a proper rewrite step $J \Rightarrow_{v,l\to r} H$. By Lemma 4.4, there are collapsings $J' \succeq M \succeq J$ and a preimage $\overline{v}$ of v in M such that $\langle \overline{v},\, l \to r\rangle$ is a redex and $J' \succeq M$ is minimal with respect to this redex. By Lemma 4.7, there is a minimally collapsing derivation $M \Rightarrow_{\overline{v},l\to r} H_1 \Rightarrow_{\mathcal{R}}^* H'$ such that $H' \succeq H$. Hence, if $J' \succeq M$ is a proper collapse step, then $G \Rightarrow_{\mathcal{R}}^* J' \succ M \Rightarrow_{\mathcal{R}} H_1 \Rightarrow_{\mathcal{R}}^* H'$ is as required. Otherwise we can omit M and obtain the minimally collapsing derivation $G \Rightarrow_{\mathcal{R}}^* J' \Rightarrow_{\mathcal{R}} H_1 \Rightarrow_{\mathcal{R}}^* H'$. □

Example 4.9. The lower half of Figure 9 shows a term graph rewrite derivation consisting of a collapse step and a proper rewrite step with the rule `f(x,x)` $\to$ `k(x)`. This derivation is transformed into the minimally collapsing derivation and the subsequent collapsing given in the upper half of the figure.

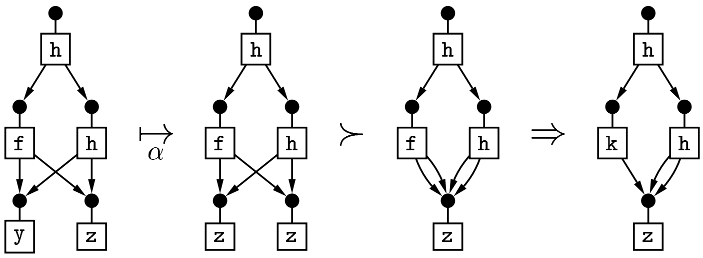


Fig. 10. The components of a term graph narrowing step

5. Term graph narrowing

Our goal is to solve term equations by transformations on term graphs. To this end we define term graph narrowing and establish a completeness result which corresponds to Hullot's result for term narrowing (Hullot 1980; Middeldorp and Hamoen 1994).

An *equation* $s = t$ is a pair of terms s and t. We are interested in solutions to such equations modulo the equational theory induced by a term rewriting system $\mathcal{R}$. That is, a *solution* of $s = t$ is a term substitution σ such that $s\sigma \leftrightarrow^*_{\mathcal{R}} t\sigma$. If such a solution exists, we say that s and t are *$\mathcal{R}$-unifiable.*

Definition 5.1. (Term graph narrowing) Let G and H be term graphs, v be a non-variable node in G, $l \to r$ be a rule[6] in $\mathcal{R}$, and α be a term graph substitution. Then there is a *narrowing step* $G \rightsquigarrow_{v,\, l\to r,\, \alpha} H$ if α^{term} is a most general unifier of l and $term_G(v)$, and

$$G\alpha \underset{c}{\succeq} G' \underset{c(v),\, l\to r}{\Longrightarrow} H$$

for some collapsing $G\alpha \succeq_c G'$. We denote such a step also by $G \rightsquigarrow_\alpha H$.

The collapsing *after* application of α is necessary to make narrowing complete[7]. For, if $l \to r$ is not left-linear, then there need not exist a step $G\alpha \Rightarrow_{v,\, l\to r} H$ even if l is unifiable with $term_G(v)$ and G is fully collapsed (see Example 5.2).

A *term graph narrowing derivation* $G \rightsquigarrow^*_\alpha H$ is either an isomorphism $G \to H$ together with the empty substitution or a non-empty sequence

$$G = G_0 \rightsquigarrow_{\alpha_1} G_1 \rightsquigarrow_{\alpha_2} \dots \rightsquigarrow_{\alpha_n} G_n = H$$

such that $\alpha = \alpha_1\alpha_2 \dots \alpha_n$.

Example 5.2. Figure 10 shows a term graph narrowing step in its three component steps. The applied term rewrite rule is $\mathtt{f(x,x)} \to \mathtt{k(x)}$ and the computed term graph substitution is $\alpha = \{\mathtt{x/Z, y/Z}\}$, where $term(Z) = \mathtt{z}$. Note that α^{term} is a most general unifier of $\mathtt{f(x,x)}$ and $\mathtt{f(y,z)}$. Since $\mathtt{f(x,x)} \to \mathtt{k(x)}$ is non-left-linear, there is no graph morphism from $\underline{\Diamond\mathtt{f(x,x)}}$ to the term graph resulting from the application of α. That is,

[6] We assume that this rule has no common variables with G. If this is not the case, then the variables in $l \to r$ are renamed into variables from $Var - Var(G)$.

[7] The definition of narrowing in (Habel and Plump 1995) is not sufficient in this respect.

the rule cannot be applied to this graph. We first have to identify the two $\mathtt{z}$-labelled edges by a collapsing.

From now on we assume that $\mathcal{R}$ contains the rule $\mathtt{x} =^? \mathtt{x} \to \mathtt{true}$, where the binary function symbol $=^?$ and the constant $\mathtt{true}$ do not occur in any other rule. A *goal* is a term of the form $s =^? t$ such that s and t do not contain $=^?$ and $\mathtt{true}$. We denote by $\Delta\mathtt{true}$ a term graph representing $\mathtt{true}$.

Example 5.3. Let $\mathcal{R}$ consist of the following rules:

$$\begin{array}{lcl@{\qquad\qquad}lcl} \mathtt{0} + \mathtt{x} & \to & \mathtt{x} & \mathtt{0} \times \mathtt{x} & \to & \mathtt{0} \\ \mathtt{S(x)} + \mathtt{y} & \to & \mathtt{S(x+y)} & \mathtt{S(x)} \times \mathtt{y} & \to & (\mathtt{x} \times \mathtt{y}) + \mathtt{y} \\ & & & \mathtt{x} =^? \mathtt{x} & \to & \mathtt{true} \end{array}$$

Suppose that we want to solve the goal $(\mathtt{z} \times \mathtt{z}) + (\mathtt{z} \times \mathtt{z}) =^? \mathtt{S(z)}$. Figure 11 shows a term graph narrowing derivation starting from a fully collapsed representation of this goal. For each narrowing step, the applied rewrite rule and the involved term substitution are given. Note that steps c,d and e are proper rewrite steps and that step f consists of a collapse step and a proper rewrite step. The derivation computes the term substitution $\{\mathtt{x/0}, \mathtt{x'/S(0)}, \mathtt{y/S(0)}, \mathtt{z/S(0)}\}$ in six steps. Restricting this substitution to the variables of the initial term graph yields the solution $\{\mathtt{z/S(0)}\}$. Solving the same goal by term narrowing requires nine steps, demonstrating that term graph narrowing speeds up the computation.

Theorem 5.4. (Soundness of narrowing) Let G be a term graph such that $term(G)$ is a goal $s =^? t$. If $G \rightsquigarrow_\alpha^* \Delta\mathtt{true}$, then α^{term} is an $\mathcal{R}$-unifier of s and t.

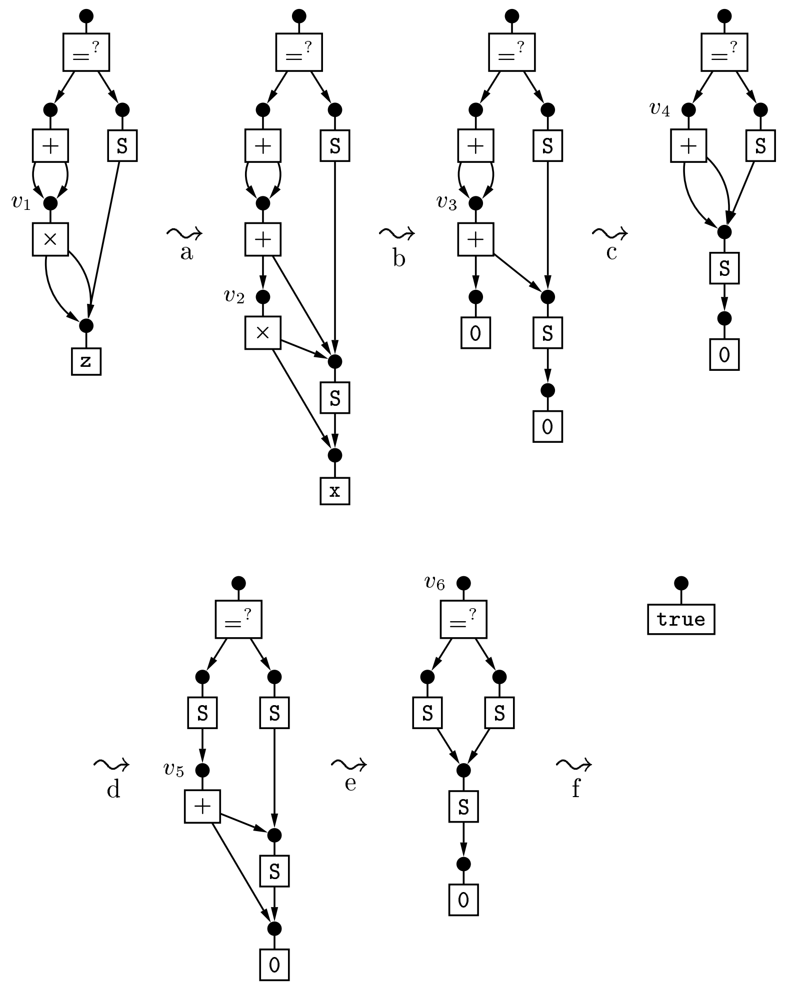
Proof. By the definition of narrowing, $M \rightsquigarrow_\beta N$ implies $M\beta \Rightarrow_\mathcal{R}^* N$. Hence, by a simple induction on the length of narrowing derivations (using Lemma 3.11), $G \rightsquigarrow_\alpha^* \Delta\mathtt{true}$ implies $G\alpha \Rightarrow_\mathcal{R}^* \Delta\mathtt{true}$. Since $term(G\alpha) = term(G)\alpha^{term} = (s\alpha^{term} =^? t\alpha^{term})$, Theorem 3.7 yields $(s\alpha^{term} =^? t\alpha^{term}) \to_\mathcal{R}^* \mathtt{true}$. This implies $s\alpha^{term} \leftrightarrow_\mathcal{R}^* t\alpha^{term}$. □

Given two term substitutions σ and τ, and a subset V of *Var*, we write $\sigma =_\mathcal{R} \tau\ [V]$ if $x\sigma \leftrightarrow_\mathcal{R}^* x\tau$ for each $x \in V$. We write $\sigma \leq_\mathcal{R} \tau\ [V]$ if there is a substitution ρ such that $\sigma\rho =_\mathcal{R} \tau\ [V]$. The *restriction* $\sigma|_V$ of a term substitution σ to a subset V of *Var* is the substitution $\{x/t \in \sigma \mid x \in V\}$. The restriction of a term graph substitution is defined analogously. A term graph substitution $\alpha = \{x_1/G_1, \ldots, x_n/G_n\}$ is *normalized* if $G_1, \ldots, G_n$ are normal forms with respect to $\Rightarrow_\mathcal{R}$.

Theorem 5.5. (Completeness of narrowing) Let $\Rightarrow_\mathcal{R}$ be convergent and G be a term graph such that $term(G)$ is a goal $s =^? t$. Then for every $\mathcal{R}$-unifier σ of s and t, there is a narrowing derivation $G \rightsquigarrow_\beta^* \Delta\mathtt{true}$ such that $\beta^{term} \leq_\mathcal{R} \sigma\ [Var(G)]$.

Proof. By Theorem 3.10.1, confluence of $\Rightarrow_\mathcal{R}$ implies confluence of $\to_\mathcal{R}$, and, by Theorem 3.9.2, termination of $\Rightarrow_\mathcal{R}$ implies normalization of $\to_\mathcal{R}$. Hence every term u has a unique normal form u_- with respect to $\to_\mathcal{R}$, and each two convertible terms have the same normal form. Let

$$\sigma' = \{x/x\sigma_- \mid x \in Dom(\sigma)\}.$$



step	node	rewrite rule	substitution
a	v_1	$\mathtt{S(x)} \times \mathtt{y} \to (\mathtt{x} \times \mathtt{y}) + \mathtt{y}$	$\{\mathtt{y/S(x)}, \mathtt{z/S(x)}\}$
b	v_2	$\mathtt{0} \times \mathtt{x}' \to \mathtt{0}$	$\{\mathtt{x/0}, \mathtt{x}'/\mathtt{S(0)}\}$
c	v_3	$\mathtt{0} + \mathtt{x} \to \mathtt{x}$	$\{\mathtt{x/S(0)}\}$
d	v_4	$\mathtt{S(x)} + \mathtt{y} \to \mathtt{S(x+y)}$	$\{\mathtt{x/0}, \mathtt{y/S(0)}\}$
e	v_5	$\mathtt{0} + \mathtt{x} \to \mathtt{x}$	$\{\mathtt{x/S(0)}\}$
f	v_6	$\mathtt{x} =^? \mathtt{x} \to \mathtt{true}$	$\{\mathtt{x/S(S(0))}\}$

Fig. 11. A term graph narrowing derivation with its rewrite rules and substitutions

Since σ is an $\mathcal{R}$-unifier of s and t, we have

$$s\sigma'_{\_} \underset{\mathcal{R}}{\overset{*}{\leftarrow}} s\sigma' \underset{\mathcal{R}}{\overset{*}{\leftarrow}} s\sigma \underset{\mathcal{R}}{\overset{*}{\leftrightarrow}} t\sigma \underset{\mathcal{R}}{\overset{*}{\rightarrow}} t\sigma' \underset{\mathcal{R}}{\overset{*}{\rightarrow}} t\sigma'_{\_}$$

and hence $s\sigma'_{\_} = t\sigma'_{\_}$. Thus

$$(s\sigma' =^? t\sigma') \underset{\mathcal{R}}{\overset{*}{\rightarrow}} (s\sigma'_{\_} =^? t\sigma'_{\_}) \underset{\mathcal{R}}{\rightarrow} \mathtt{true}.$$

Now consider the term graph substitution

$$\alpha = \{x/\nabla x\sigma' \mid x \in \mathit{Dom}(\sigma')\},$$

where $\nabla x\sigma'$ is a fully collapsed term graph representing $x\sigma'$. By Theorem 3.9.1, α is normalized. Let $G\alpha \Rightarrow^*_{\mathcal{R}} H$ be a rewrite derivation such that H is the normal form of $G\alpha$. Then, by Theorem 3.9.1, $\mathit{term}(H)$ is the normal form of $\mathit{term}(G\alpha)$. Moreover, we have $\mathit{term}(G\alpha) = (s\alpha^{\mathit{term}} =^? t\alpha^{\mathit{term}}) = (s\sigma' =^? t\sigma')$ since $\alpha^{\mathit{term}} = \sigma'$. Hence $\mathit{term}(H) = \mathtt{true}$ and $H \cong \Delta\mathtt{true}$. Let now $V = \mathit{Var}(G) \cup \mathit{Dom}(\alpha)$. By the Lifting Lemma given in Section 6, the rewrite derivation $G\alpha \Rightarrow^*_{\mathcal{R}} \Delta\mathtt{true}$ can be transformed into a narrowing derivation $G \rightsquigarrow^*_{\beta} H'$ such that there is a term graph substitution γ with $H'\gamma \succeq \Delta\mathtt{true}$ and $(\beta\gamma)|_V = \alpha$. Since $\mathit{term}(H')$ is not a goal, we have $H' \cong \Delta\mathtt{true}$ and

$$G \underset{\beta}{\overset{*}{\rightsquigarrow}} \Delta\mathtt{true}.$$

From $(\beta\gamma)|_V = \alpha$ we obtain with Lemma 2.11

$$\begin{aligned} (\beta^{\mathit{term}}\gamma^{\mathit{term}})|_V &= (\beta\gamma)^{\mathit{term}}|_V \\ &= ((\beta\gamma)|_V)^{\mathit{term}} \\ &= \alpha^{\mathit{term}} \\ &= \sigma'. \end{aligned}$$

Then $\beta^{\mathit{term}}\gamma^{\mathit{term}} =_{\mathcal{R}} \sigma\ [V]$ since $\sigma' =_{\mathcal{R}} \sigma\ [V]$. Hence $\beta^{\mathit{term}} \leq_{\mathcal{R}} \sigma\ [V]$, and the proposition follows with $\mathit{Var}(G) \subseteq V$. □

By Theorem 3.10.2, $\Rightarrow_{\mathcal{R}}$ is convergent whenever $\rightarrow_{\mathcal{R}}$ is. Hence we have the following corollary.

Corollary 5.6. Term graph narrowing is complete for every convergent term rewriting system.

Thus, the comprehensive theory developed in the of area term rewriting to prove termination and confluence (see Dershowitz and Jouannaud (1990) and Klop (1992) for overviews) can be used to show that term graph narrowing is complete for a given system $\mathcal{R}$. Moreover, a terminating term graph rewrite relation $\Rightarrow_{\mathcal{R}}$ can be tested for confluence by analyzing *critical pairs* of term graph rewrite steps (Plump 1994). This method is applicable even in those cases where $\Rightarrow_{\mathcal{R}}$ is terminating although $\rightarrow_{\mathcal{R}}$ is non-terminating.

Inspecting the proof of Theorem 5.5 shows that termination of $\Rightarrow_{\mathcal{R}}$ can be relaxed to normalization. Hence we can strengthen the completeness result as follows.

Theorem 5.7. Term graph narrowing is complete whenever term graph rewriting is normalizing and confluent.

As in the case of term narrowing, neither normalization nor confluence of rewriting

$$\begin{tikzcd} G' \arrow[r, squiggly, "*", "\beta"'] \arrow[dd, "\alpha"'] & H' \arrow[d, "\gamma"] \\ & H'\gamma \arrow[d] \\ G \arrow[r, Rightarrow, "*", "\mathcal{R}"'] & H \end{tikzcd}$$

Fig. 12. Lifting of a rewrite derivation

can be dropped. For example, the system $\mathcal{R} = \{\mathtt{a} \to \mathtt{b},\, \mathtt{a} \to \mathtt{c}\}$ is terminating under both term and term graph rewriting, but not confluent. Although the empty substitution is an $\mathcal{R}$-unifier of the terms $\mathtt{b}$ and $\mathtt{c}$, there is no narrowing step starting from the goal $\mathtt{b} =^? \mathtt{c}$. A system showing the necessity of normalization is $\mathcal{R} = \{\mathtt{a} \to \mathtt{f(a)}\}$. Here $\to_\mathcal{R}$ and $\Rightarrow_\mathcal{R}$ are confluent (where for $\Rightarrow_\mathcal{R}$ we must require that there are no function symbols of arity greater than 1), and $\{\mathtt{x/a}\}$ is an $\mathcal{R}$-unifier of $\mathtt{x}$ and $\mathtt{f(x)}$. However, there is no narrowing step starting from $\mathtt{x} =^? \mathtt{f(x)}$.

6. The Lifting Lemma

In this section we establish the main tool of the above completeness proof: the lifting of term graph rewrite derivations to term graph narrowing derivations. Lifting proceeds in two steps: at first the given rewrite derivation is transformed into a minimally collapsing rewrite derivation with a subsequent collapsing, and then this derivation is directly lifted to a narrowing derivation.

Lemma 6.1. (Lifting Lemma) Let $G \Rightarrow^*_\mathcal{R} H$ be a rewrite derivation and G' be a term graph such that $G'\alpha = G$ for some normalized substitution α. Moreover, let V be a finite subset of *Var* such that $\mathit{Var}(G') \cup \mathit{Dom}(\alpha) \subseteq V$. Then there is a narrowing derivation $G' \rightsquigarrow^*_\beta H'$ and a normalized substitution γ such that $H'\gamma \succeq H$ and $(\beta\gamma)|_V = \alpha$.

Proof of the Lifting Lemma. By Theorem 4.8 (transformation of rewrite derivations), there is a minimally collapsing rewrite derivation $G \Rightarrow^*_\mathcal{R} \overline{H}$ such that $\overline{H} \succeq H$. Hence, by Lemma 6.4 given below (lifting of minimally collapsing rewrite derivations), there is a narrowing derivation $G' \rightsquigarrow^*_\beta H'$ and a normalized substitution γ such that $H'\gamma = \overline{H}$ and $(\beta\gamma)|_V = \alpha$. □

The following example shows that the "vertical collapsing" $H'\gamma \succeq H$ (see Figure 12) cannot be omitted.

Example 6.2. The bottom row of Figure 13 contains a single collapse step $G \succ H$. The term graphs G' and G on the left of the top and the bottom row, respectively, are related by a term graph substitution $\alpha = \{\mathtt{x/A}, \mathtt{y/A}\}$ with $\mathit{term}(A) = \mathtt{a}$. Since the rewrite step $G \Rightarrow H$ is not proper, it clearly has to be lifted to a narrowing derivation $G' \rightsquigarrow^*_\emptyset H'$ of length 0. But then H' cannot directly be transformed into H by applying a substitution. However, we obtain H from $H'\alpha$ by a collapsing.

The two-step lifting procedure used in the proof of the Lifting Lemma is demonstrated in the next example.

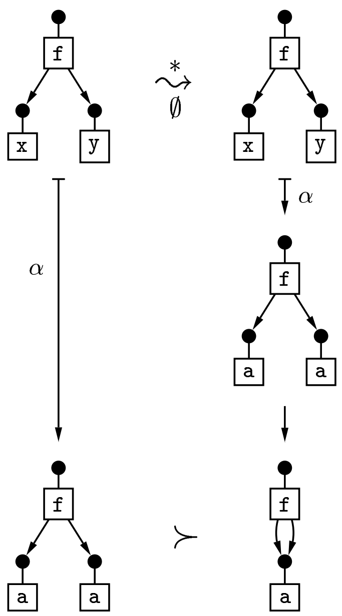


Fig. 13. The need for "vertical collapsing"

Example 6.3. A term graph rewrite derivation consisting of a collapse step and a proper rewrite step with rule $\mathtt{f(x,x)} \to \mathtt{k(x)}$ is shown in the bottom row of Figure 14. The initial term graph of this derivation is obtained from the left term graph of the top row by a term graph substitution $\alpha = \{\mathtt{y/A}, \mathtt{z/A}\}$ with $term(A) = \mathtt{g(a)}$. The derivation on the bottom is first transformed into the minimally collapsing rewrite derivation in the middle row, which then is lifted to the narrowing derivation in the top row. The first narrowing step is presented in form of its components: application of a term graph substitution, collapsing, and application of a rewrite rule. The term graph substitution computed by this step is $\beta_1 = \{\mathtt{x/Z}, \mathtt{y/Z}\}$ with $term(Z) = \mathtt{z}$. The second narrowing step is a rewrite step with rule $\mathtt{f(x,x)} \to \mathtt{k(x)}$ and associated substitution $\beta_2 = \{\mathtt{x/Z}\}$. The resulting term graph can be transformed into the final graph of the minimally collapsing derivation by an application of $\gamma = \{\mathtt{z/A}\}$ (with $term(A) = \mathtt{g(a)}$). Note that we obtain the given substitution α if we restrict $\beta_1\beta_2\gamma$ to the variables of the top left graph.

In the proofs of the following lemmas, we have to take care of the variables introduced by substitutions. Given a term graph substitution $\alpha = \{x_1/G_1, \ldots, x_n/G_n\}$, let $I(\alpha)$ be the set $\bigcup_{i=1}^{n} Var(G_i)$. For a term substitution σ, $I(\sigma)$ is defined analogously.

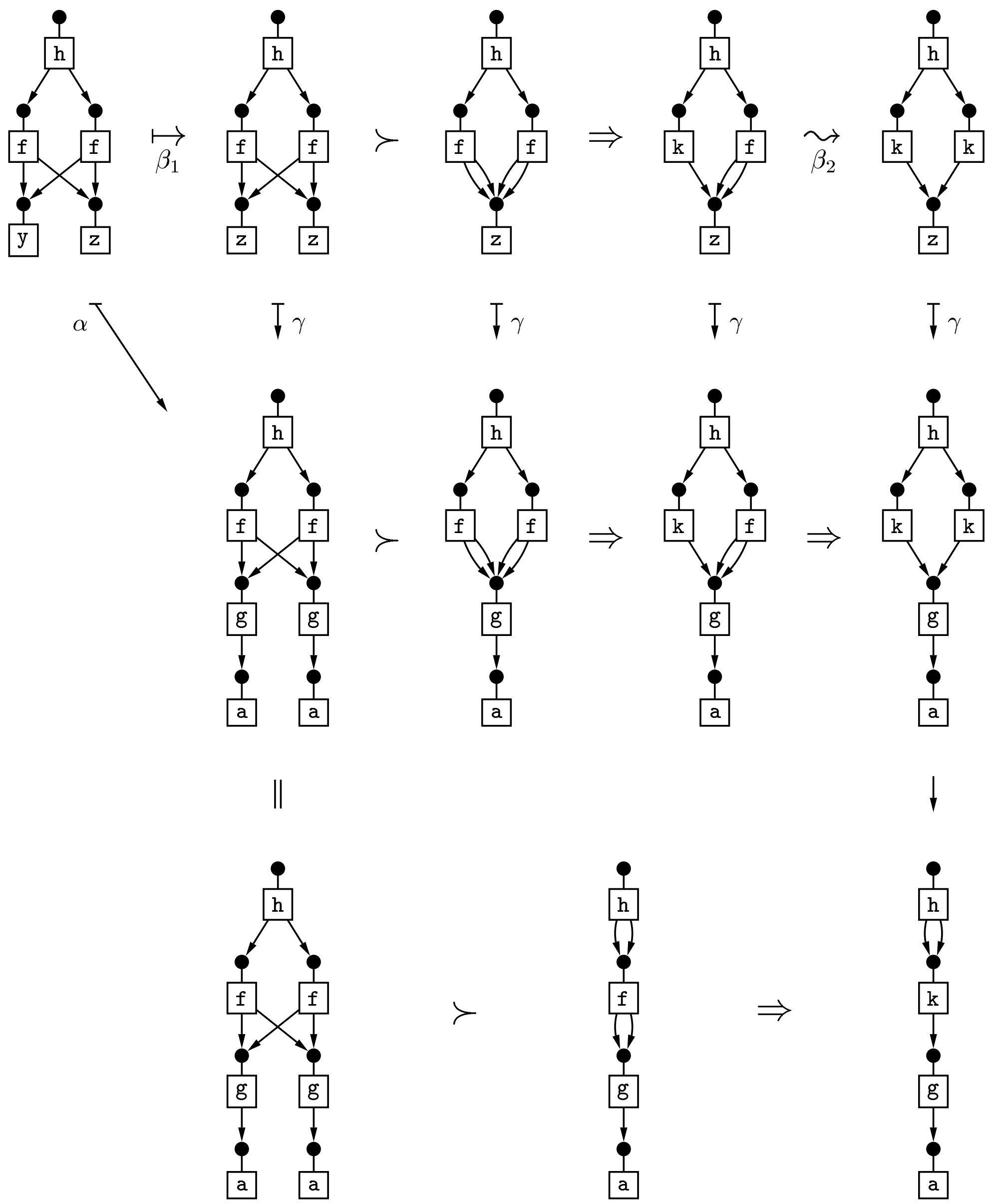


Fig. 14. Lifting of a rewrite derivation

Lemma 6.4. (Lifting of minimally collapsing rewrite derivations) Let $G \Rightarrow_{\mathcal{R}}^{*} H$ be a minimally collapsing rewrite derivation and G' be a term graph such that $G'\alpha = G$ for some normalized substitution α. Moreover, let $V \subseteq \mathit{Var}$ be a finite set of variables such that $\mathit{Var}(G') \cup \mathit{Dom}(\alpha) \subseteq V$. Then there is a narrowing derivation $G' \rightsquigarrow_{\beta}^{*} H'$ and a normalized substitution γ such that $H'\gamma = H$ and $(\beta\gamma)|_V = \alpha$.

Proof. By induction on the number n of proper rewrite steps in $G \Rightarrow_{\mathcal{R}}^{*} H$.
Induction base. If $G \Rightarrow_{\mathcal{R}}^{*} H$ contains no proper rewrite steps, then $G \cong H$. Hence a narrowing derivation $G' \rightsquigarrow_{\emptyset}^{*} H'$ of length 0 and $\gamma = \alpha$ have the desired properties.
Induction step. Let $G \Rightarrow_{\mathcal{R}}^{*} H$ be a minimally collapsing rewrite derivation with $n+1$ proper rewrite steps ($n \geq 0$). Decompose this derivation into two minimally collapsing rewrite derivations $G \Rightarrow_{\mathcal{R}}^{*} H_1 \Rightarrow_{\mathcal{R}}^{*} H$ containing one and n proper rewrite steps, respectively. By Lemma 6.5 presented below, there is a narrowing step $G' \rightsquigarrow_{\beta_1} H_1'$ and a normalized substitution γ_1 such that $H_1'\gamma_1 = H_1$ and $(\beta_1\gamma_1)|_V = \alpha$. Let $V_1 = (V - \mathit{Dom}(\beta_1)) \cup I(\beta_1)$ and $\gamma_1' = \gamma_1|_{V_1}$. Then $\mathit{Var}(H_1') \subseteq \mathit{Var}(G'\beta_1) \subseteq V_1$ and $\mathit{Dom}(\gamma_1') \subseteq V_1$. Moreover, we have $H_1'\gamma_1 = H_1'\gamma_1'$. Hence, by induction hypothesis, there is a narrowing derivation $H_1' \rightsquigarrow_{\beta_2}^{*} H'$ and a normalized substitution γ such that $H'\gamma = H$ and $(\beta_2\gamma)|_{V_1} = \gamma_1'$. Combining the narrowing derivations, we obtain a narrowing derivation $H \rightsquigarrow_{\beta_1\beta_2}^{*} H'$ and a normalized substitution γ such that $H'\gamma = H$. It remains to show that $(\beta_1\beta_2\gamma)|_V = \alpha$. Since $(V - \mathit{Dom}(\beta_1)) \cup I(\beta_1) = V_1$ and $(\beta_2\gamma)|_{V_1} = \gamma_1' = \gamma_1|_{V_1}$, we have $(\beta_1\beta_2\gamma)|_V = (\beta_1\gamma_1)|_V = \alpha$. □

The next lemma shows that a proper rewrite step together with a preceding collapsing can be lifted to a narrowing step, provided the collapsing is minimal. In the proof, we use a particular factorization of substitutions (Lemma 6.6) and exploit that minimal collapsings can be lifted over substitutions (Lemma 6.9).

Lemma 6.5. Let $G \succeq M \Rightarrow_{v,l\to r} H$ be a minimally collapsing rewrite derivation with one proper rewrite step, and G' be a term graph such that $G'\alpha = G$ for some normalized substitution α. Moreover, let $V \subseteq \mathit{Var}$ be a finite set of variables such that $\mathit{Var}(G') \cup \mathit{Dom}(\alpha) \subseteq V$. Then there is a narrowing step $G' \rightsquigarrow_{\beta} H'$ and a normalized substitution γ such that $H'\gamma = H$ and $(\beta\gamma)|_V = \alpha$ (see the following diagram).

$$\begin{tikzcd}
G' \arrow[r, mapsto, "\beta"'] \arrow[dr, "\alpha"'] & G'\beta \arrow[d, mapsto, "\gamma"] & \succeq & M' \arrow[r, Rightarrow, "v,l\to r"'] \arrow[d, mapsto, "\gamma"] & H' \arrow[d, mapsto, "\gamma"] \\
 & G & \succeq & M \arrow[r, Rightarrow, "v,l\to r"'] & H
\end{tikzcd}$$

Proof. Without loss of generality, we may assume $\mathit{Var}(l) \cap V = \emptyset$ (otherwise the variables of l are renamed). The node v has a unique preimage in G, say u, and we have $\mathit{term}_G(u) = \mathit{term}_M(v) = l\sigma$ for some term substitution σ. Since α is normalized, u is a non-variable node in G'. By Lemma 6.6 presented below, there exist term graph substitutions β and γ such that

(1) β^{term} is a most general unifier of $\mathit{term}_{G'}(u)$ and l,
(2) $(\beta\gamma)|_V = \alpha$, and
(3) γ is normalized.

By Lemma 6.9 given below, there exists a collapsing $G'\beta \succeq M'$ such that $M'\gamma = M$. Node v belongs to M' because γ is normalized. Moreover, the collapsing sends u to v. Since β^{term} is a unifier of $term_{G'}(u)$ and l, $term_{M'}(v) = term_{G'\beta}(u) = term_{G'}(u)\beta^{term} = l\beta^{term}$. Hence $red(\underline{\Diamond l}) \subseteq \underline{M'}$ for the redex morphism $red\colon \underline{\Diamond l} \to M$. With the Restriction Lemma 3.12 we obtain a proper rewrite step $M' \Rightarrow_{v,l\to r} H'$ such that $H'\gamma = H$. Thus, there is a narrowing step $G' \rightsquigarrow_\beta H'$ and a normalized substitution γ such that $H'\gamma = H$ and $(\beta\gamma)|_V = \alpha$. $\square$

Below we present the two lemmas used in the proof of Lemma 6.5. We begin with the factorization of the term graph substitution α into a most general substitution β and a remaining substitution γ such that $G'\beta$ contains a preredex $\langle v,\, l \to r\rangle$.

Lemma 6.6. (Factorization of substitutions) Let $G'\alpha = G$ for some normalized term graph substitution α and v be a node in G' such that $term_G(v) = l\sigma$ for some rule $l \to r$ in $\mathcal{R}$ and term substitution σ. Moreover, let V be a finite subset of Var such that $Var(G') \cup Dom(\alpha) \subseteq V$. Then there exist term graph substitutions β and γ such that

(1) β^{term} is a most general unifier of $term_{G'}(v)$ and l,
(2) $(\beta\gamma)|_V = \alpha$, and
(3) γ is normalized.

Proof. Without loss of generality, we may assume $V \cap Var(l) = \emptyset$. By Lemma 2.10, $term_{G'}(v)\alpha^{term} = term_{G'\alpha}(v) = term_G(v) = l\sigma$ for some term substitution σ with $Dom(\sigma) \subseteq Var(l)$. Let $\mu = \alpha^{term} \cup \sigma$. Then we have $term_{G'}(v)\mu = term_{G'}(v)\alpha^{term} = l\sigma = l\mu$, so $term_{G'}(v)$ and l are unifiable. Let τ be an idempotent most general unifier of $term_{G'}(v)$ and l. Then there exists a term substitution ρ such that $\tau\rho = \mu$. By idempotence of τ, $Dom(\tau) \cup I(\tau) = Var(term_{G'}(v)) \cup Var(l)$.
Now consider the term graph substitution $\beta = \{x/\nabla x\tau \mid x \in Dom(\tau)\}$, where $\nabla x\tau$ is a fully collapsed term graph representing $x\tau$. Then β is idempotent and $Dom(\beta) \cup I(\beta) = Var(G'[v]) \cup Var(l)$. Let $V' = (V - Dom(\beta)) \cup I(\beta)$ and $\gamma = \{x/\nabla x\rho \mid x \in Dom(\rho)|_{V'}\}$. Then $Var(G'\beta) = (Var(G') - Dom(\beta)) \cup I(\beta|_{Var(G')}) \subseteq V'$ and $Dom(\gamma) \subseteq V'$. Therefore, $Var(G'\beta) \cup Dom(\gamma) \subseteq V'$. Next we show that $(\beta\gamma)|_V = \alpha$. Since $\beta^{term} = \tau|_V$ and $\gamma^{term} = \rho|_{V'}$, we obtain $(\beta^{term}\gamma^{term})|_V = \tau\rho|_V$. Moreover, $\tau\rho = \mu$ and $\mu|_V = \alpha^{term}$. Thus, $(\beta^{term}\gamma^{term})_V = \alpha^{term}$. Since α is normalized, $\alpha = \{x/\nabla x\alpha^{term} \mid x \in Dom(\alpha)\}$. Consequently, $(\beta\gamma)|_V = \alpha$. It remains to verify that γ is normalized. Let $V'' = (V - Dom(\beta)) \cup I(\beta|_V)$. Since α is normalized and $(\beta\gamma)|_V = \alpha$, $\beta|_V$ and $\gamma|_{V''}$ are normalized. We claim $I(\beta) \subseteq V''$ and hence $V'' = V'$. Recall that $I(\beta) \subseteq Var(G'[v]) \cup Var(l)$. Let $x \in I(\beta)$. By idempotence of β, $x \notin Dom(\beta)$. If $x \in Var(G'[v])$, then $x \in V - Dom(\beta) \subseteq V''$. If $x \in Var(l)$, then $x \in Var(l\tau) = Var(term_{G'}(v)\tau) = Var(G'[v]\beta)$ and thus $x \in I(\beta|_V) \subseteq V''$. So $I(\beta) \subseteq V''$ and $V'' = V'$. Since $Dom(\gamma) \subseteq V'$, we conclude that γ is normalized. $\square$

Example 6.7. Let G' and G be the term graphs on the left and the right in Figure 15. We have $G'\alpha = G$ for the term graph substitution $\alpha = \{\mathtt{z1}/\mathtt{A}, \mathtt{z2}/\mathtt{A}\}$ with $term(A) = \mathtt{g}(\mathtt{a})$. Suppose that $\mathtt{f}(\mathtt{x}, \mathtt{g}(\mathtt{y}))$ is the left-hand side of some rewrite rule. Then there are two (pre)redexes for this rule in G, but no preredex in G'. So α creates two preredexes, but is not a most general substitution with this property. A term graph substitution of the latter kind is $\beta = \{\mathtt{z1}/\mathtt{B}_1, \mathtt{z2}/\mathtt{B}_2\}$ with $term(B_1) = \mathtt{x}$ and $term(B_2) = \mathtt{g}(\mathtt{y})$. That is, β^{term} is a

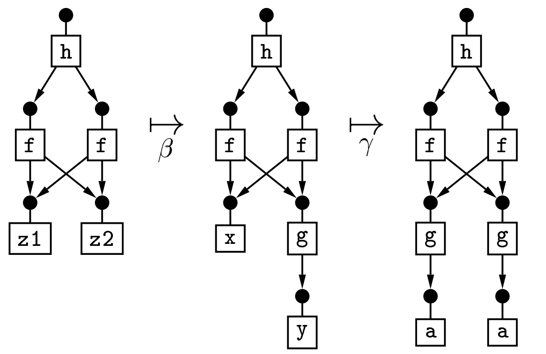


Fig. 15. Factorization of a substitution

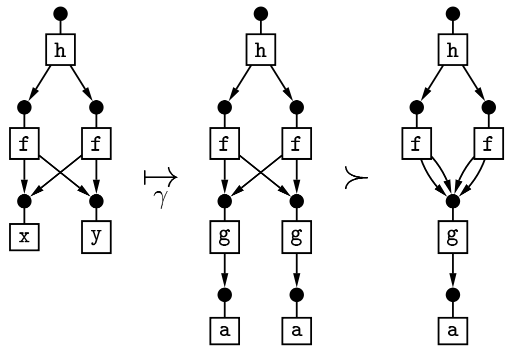


Fig. 16. A non-liftable collapse step

most general unifier of $\mathtt{f(z1,z2)}$ and the left-hand side $\mathtt{f(x,g(y))}$. By composing β with $\gamma = \{\mathtt{x/A}, \mathtt{y/C}\}$, where $\mathit{term}(C) = \mathtt{a}$, and restricting the result to $\mathit{Var}(G')$, we obtain the given substitution α.

Below we consider minimal collapsings $G'\gamma \succeq H$ that can be "lifted" over the term graph substitution γ to the term graph G'. That is, there has to be a collapsing $G' \succeq H'$ such that $H'\gamma = H$. Such a lifting fails if the preredex in $G'\gamma$ is created by the substitution γ. This is demonstrated by the following example.

Example 6.8. Figure 16 shows a minimal collapsing $G'\gamma \succ H$ together with a preceding substitution application. While $G'\gamma$ contains a preredex for the rule $\mathtt{f(x,x)} \to \mathtt{h(x)}$, there is no preredex in G'. That is, the preredex is created by the substitution. As a consequence, there does not exist a collapsing $G' \succeq G''$ such that G'' can be transformed into H by applying a substitution.

Lemma 6.9. (Lifting of mimimal collapsings) Let $G \succeq_c H$ be a collapsing that is minimal with respect to a redex $\langle v, l \to r\rangle$ in H, and let G' be a term graph such that $G'\gamma = G$ for some normalized term graph substitution γ. Let v' be the unique preimage of v in G. If $\langle v', l \to r\rangle$ is a preredex in G', then there is a collapsing $G' \succeq_{c'} H'$ such

$$\begin{tikzcd} \underline{G'} \arrow[r, "\underline{c'}"] \arrow[d] & \underline{H'} \arrow[d] \\ G'\gamma \arrow[r, "\overline{c}"] \arrow[d, "id"'] & H'\gamma \arrow[d, "h"] \\ G \arrow[r, "c"] & H \end{tikzcd}$$

Fig. 17. Lifting of a minimal collapsing

that $H'\gamma = H$. Moreover, $\langle c'(v'), l \to r\rangle$ is a redex in H' and $G' \succeq_{c'} H'$ is minimal with respect to this redex.

Proof. The collapse step $G' \succeq_{c'} H'$ with $\underline{H'} \subseteq H$ is constructed as follows. Let $\underline{c'}: \underline{G'} \to c(\underline{G'})$ be the restriction of c to $\underline{G'}$ and $c(\underline{G'})$, let H' be obtained from $c(\underline{G'})$ by appending an edge e_u with label $term_{G'}(u)$ to each image of a variable node u in G', and let $c': G' \to H'$ be the extension of $\underline{c'}$ to G' with $c'(e) = e_u$ for each variable edge e with result node u. For H' to be well-defined, c must not identify nodes in G' that represent different terms in G'. The latter holds true for the following reasons: Since $G \succeq_c H$ is minimal, c identifies two distinct nodes v_1 and v_2 only if they are reachable from two nodes v_1' and v_2' corresponding to two distinct nodes in the tree representation of l that represent the same variable. Since $\langle v', l \to r\rangle$ is a preredex in G', we have $term_{G'}(v') = l\sigma$ for some term substitution σ. Hence v_1' and v_2' belong to G' and satisfy $term_{G'}(v_1') = term_{G'}(v_2')$. Thus, each two nodes in G' that are identified by c represent the same term in G'. So H' is well-defined. Moreover, we have $\underline{H'} \subseteq H$.
By the definition of substitution, for every variable node u in G', there is an isomorphism $G'\gamma[u] \to H'\gamma[c(u)]$. Thus, $\underline{c'}: \underline{G'} \to \underline{H'}$ can be extended to a surjective morphism $\overline{c}: G'\gamma \to H'\gamma$. Moreover, there is a morphism $h: H'\gamma \to H$: Firstly, there is an inclusion morphism $\underline{H'} \to H$. Secondly, for every variable node w in H' and every preimage u of w in G', there is an isomorphism $H'\gamma[w] \to G'\gamma[u]$. Moreover, the restriction of c to $G'\gamma[u]$ defines a morphism $G'\gamma[u] \to H$. Since γ is normalized, $G'\gamma[u]$ is fully collapsed and $G'\gamma[u] \to H$ is injective. The composition of the morphisms yields an injective morphism $H'\gamma[w] \to H$. Finally, we have $H'\gamma[w_1] \cap H'\gamma[w_2] = \emptyset$ for distinct variable nodes w_1, w_2 in H'. Thus, there is a morphism $h: H'\gamma \to H$[8] (see Figure 17).
The morphism h is surjective since $h \circ \overline{c} = c$ and because c is surjective. To see that h is injective, let b and b' be distinct nodes in $H'\gamma$. Three cases have to be considered.
Case 1. b, b' are in $\underline{H'}$. Then $h(b) \neq h(b')$ because $\underline{H'} \to H$ is injective.
Case 2. b is in $H'\gamma[w]$ and b' is in $H'\gamma[w']$, for some variable nodes w, w' in H'. If $w = w'$, then $h(b) \neq h(b')$ because $H'\gamma[w] \to H$ is injective. Otherwise, let a, a', u, u' be preimages of b, b', w, w' in $G'\gamma$, respectively. Then u, u' are variable nodes in G', $u \geq_{G'\gamma} a$, and $u' \geq_{G'\gamma} a'$. Since $w \neq w'$, we have $c(u) \neq c(u')$. Minimality of c implies $c(a) \neq c(a')$. Hence $h(b) \neq h(b')$.

[8] Without loss of generality, we may assume $Var(H') = Var(G') \subseteq Dom(\gamma)$. For, if this is not the case, we can consider the elements of $Var(H') - Dom(\gamma)$ as constants.

Case 3. b is a non-variable node in $\underline{H'}$ and b' is in $H'\gamma[w']$ for some variable node w' in H'. Then $h(b) \neq h(b')$ follows by reasons similar to those given above for the well-definedness of H'.

Thus h is an isomorphism, and we may assume without loss of generality $H'\gamma = H$. Moreover, from our construction it is clear that $\langle c'(v'), l \to r\rangle$ is a redex in H' and that $G' \succeq_{c'} H'$ is minimal with respect to this redex. □

7. Conclusion

We have introduced term graph narrowing as a mechanism for solving equations by transformations on term graphs. The advantage of term graph narrowing over conventional narrowing is that common subterms can be shared. Sharing saves not only space but also time since repeated computations can be avoided.

We have shown that term graph narrowing is a complete equation solving method for all term rewriting systems over which term graph rewriting is normalizing and confluent. This includes all convergent term rewriting systems. To achieve completeness, narrowing steps have to allow a collapsing between the substitution application and the rewrite step.

Our completeness proof is based on two results. On the one hand, we have shown that minimally collapsing rewrite derivations can be lifted to narrowing derivations, where minimally collapsing derivations contain only collapse steps that are necessary to enable applications of non-left-linear rewrite rules. On the other hand, we have established a normal form result for term graph rewrite derivations, showing that every derivation can be transformed into a minimally collapsing derivation together with a subsequent collapsing.

Our results suggest to consider *minimally collapsing term graph narrowing* as a restricted form of term graph narrowing in which narrowing steps contain only collapse steps that are minimal with respect to the rewrite steps. From our proofs it is easy to see that minimally collapsing narrowing is in the same sense complete as general term graph narrowing.

Future work should adapt narrowing strategies known from term narrowing to the graph world, in order to reduce the search space of term graph narrowing. It will be interesting to investigate the completeness of such strategies. In particular, we expect that *basic narrowing* is still complete. In basic narrowing, narrowing steps are not applied at subterms created by substitutions. Moreover, one should consider the completeness of particular strategies for the case that termination is relaxed to normalization (see Middeldorp and Hamoen (1994) for the completeness of basic term narrowing under various conditions).

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