1. Symbolise the following:
   (a) “Snakes are reptiles.”
   (b) “Snakes are not all poisonous.”
   (c) “Children are present.”
   (d) “Executives all have secretaries.”
   (e) “Only executives have secretaries.”
   (f) “Only community charge payers may vote in local elections.”
   (g) “Employees may use only the goods lift.”
   (h) “Only employees may use the goods lift.”
   (i) “All that glitters is not gold.”
   (j) “All estate agents are not the same.”
   (k) “Not all estate agents are the same.”
   (l) “None but the brave deserve the fair.”
   (m) “Not every visitor stayed for dinner.”
   (n) “Not any visitor stayed for dinner.”
   (o) “Nothing in the house escaped the children.”
   (p) “Some students are both intelligent and hardworking.”
   (q) “No coat is waterproof unless it has been specially treated.”
   (r) “Some medicines are dangerous only if taken in excessive amounts.”
   (s) “All fruits and vegetables are wholesome and nourishing.”
   (t) “Everything enjoyable is either immoral, illegal, or fattening.”
   (u) “A lecturer is a good teacher if, and only if, he is both well-informed and entertaining.”
   (v) “Only university lecturers and firemen are both vastly underpaid and indispensable.”
   (w) “Not every actor who is famous is talented.”
   (x) “It simply isn't true that every watch will keep good time if and only if it is wound regularly and not abused.”
   (y) “Not every person who talks a great deal has a great deal to say.”
   (z) “No car that is over ten years old will be mended if it is severely damaged.”
2. Symbolise the following predicates about the nature of elephants.
   (a) "Any elephant is attractive, if it is neat and well-groomed."
   (b) "Some elephants are gentle and have been well trained."
   (c) "Some elephants are gentle only if they have been well groomed by every student."
   (d) "Some elephants called Jumbo are gentle if they have been well trained."
   (e) "Any elephant is gentle that has been well trained."
   (f) "Any elephant called Jumbo that is gentle has been well trained."
   (g) "No elephant is gentle unless it has been well trained."
   (h) "Any elephant is gentle if it has been well trained."
   (i) "Any elephant has been well trained if it is gentle."
   (j) "Any elephant is gentle if and only if it has been well trained."
   (k) "Gentle elephants have all been well trained."
   (l) "All elephants are called either Jumbo or Dumbo."
   (m) "Every student must ride to graduation on an elephant."

3. Symbolise the following predicates about the nature of time.
   (a) "Every instant of time follows another."
   (b) "If two instants of time are not identical, then one follows the other."
   (c) "Time has no beginning."
   (d) "Time has no end."
   (e) "One instant is after a second instant only if the second is before the first."

4. Symbolise the following predicates about the numbers.
   (a) "There's a number between 3 and 5."
   (b) "Given any number there's a smaller one."
   (c) "There's no biggest number."
   (d) "Addition is commutative."
   (e) "There are two numbers which are such that their product is less than their sum."
   (f) "No cube can be expressed as the sum of two other cubes (unless at least one of the three numbers is zero)."
   (g) "If \( n > 2 \), the equation \( x^n + y^n = z^n \) cannot be solved in integers \( x, y, z \) all non-zero."

5. Identify the free and bound variables in each of the following.
   (a) \( (\forall x : T \cdot A(x)) \Rightarrow (\exists y : U \cdot B(x, y)) \)
   (b) \( A(x, y) \land (\exists x : T \cdot B(y)) \Rightarrow (\forall y : U; z : V \cdot C(x, y, z)) \)
   (c) \( (\forall x : T \cdot \exists y : U \cdot A(y, x) \land (\forall y : V \cdot C(y))) \Rightarrow B(x, y) \)
   (d) \( \forall x : T; y : U \cdot A(z) = B(z) \)
   (e) \( A(x) \Rightarrow (B(y) \Rightarrow (\exists x : T \cdot C(y)) \Rightarrow (\forall y : U \cdot D(x)))) \)

6. In each of the following, perform the intended substitutions in the corresponding predicates in the last exercise, if the substitutions are legal.
   (a) Substitute \( f(x, z) \) for \( x \).
   (b) Substitute \( z \) for \( x \), and \( g(y, z) \) for \( y \).
   (c) Substitute \( y \) for \( x \), and \( f(x, y) \) for \( y \).
   (d) Substitute \( x \) for \( z \).
   (e) Substitute \( f(y) \) for \( x \), and \( f(y) \) for \( y \).
7. Express the following as faithfully as possible, using a predicate that starts with a universal or existential quantifier.
   (a) Every noise appals me.
   (b) Something wicked this way comes.
   (c) I have a strange affirmity.
   (d) Their candles are all out.
   (e) He has no children.
   (f) Murders have been performed.
   (g) \( x \) is a tale told by an idiot.
   (h) None of woman born shall harm Macbeth.

8. Formalise the following propositions. For example, “everyone is married” would be formalised as
   \[ \forall x : \text{Person} \implies \exists y : \text{Person} \implies \text{married}(x, y) \]
   (a) There is someone who is married to everyone else.
   (b) For every integer \( x \) there is an integer \( y \) such that the sum of \( x \) and \( y \) is 0.
   (c) There is a number \( y \), such that for every number \( x \), the sum of \( x \) and \( y \) is 0.
   (d) No \( x \) is less than 0.

9. Find a predicate \( p \) in which \( x \) occurs, and for which \( \forall x : \mathbb{N} \implies p \) and \( \exists x : \mathbb{N} \implies p \) are both false.
10. Find a predicate \( p \) in which \( x \) occurs, and for which \( \forall x : \mathbb{N} \implies p \) and \( \exists x : \mathbb{N} \implies p \) are both true.

11. Let the following be defined:
    \begin{align*}
    N(x) & \quad \text{“}x\text{ is nonnegative”} \\
    E(x) & \quad \text{“}x\text{ is even”} \\
    O(x) & \quad \text{“}x\text{ is odd”} \\
    P(x) & \quad \text{“}x\text{ is prime”}
    \end{align*}
    Formalise the following:
    (a) There is an even integer.
    (b) Every integer is either even or odd.
    (c) All prime integers are nonnegative.
    (d) The only even prime is two.
    (e) There is one and only one even prime.
    (f) Not all integers are odd.
    (g) Not all primes are odd.
    (h) If an integer isn’t even, then it’s odd.

12. Give a counterexample to the assertion
    \[ (\exists x : \mathbb{N} \implies p) \land (\exists x : \mathbb{N} \implies q) \vdash \exists x : \mathbb{N} \implies p \land q \]

13. Give a counterexample to the assertion
    \[ \forall x : \mathbb{N} \implies p \lor q \vdash (\forall x : \mathbb{N} \implies p) \lor (\forall x : \mathbb{N} \implies q) \]

14. Let \( A \) be a two-dimensional integer array with 20 rows (indexed from 1 to 20), and 30 columns (indexed from 1 to 30). Using the predicate calculus, make the following assertions:
    (a) All entries of \( A \) are nonnegative.
    (b) All entries of the 4th and 15th rows are positive.
    (c) Some entries of \( A \) are zero.
    (d) The entries of \( A \) are sorted into row-major order; that is, the entries are in order within rows, and every entry of the \( i \)th row is less than or equal to every entry of the \( (i+1) \)th row.
15. There is another quantifier

\[ \exists_1 x : S \cdot p \]

which means "there is a unique \( x \) in \( S \), such that \( p \) holds".

(a) Define this quantifier in terms of universal and existential quantification.

(b) Universal quantification is a generalisation of conjunction; existential quantification is a generalisation of disjunction. Of what combinator of propositions is the unique quantifier a generalisation?

16. Prove the following laws about quantifiers:

(a) \((\exists x : S \cdot p \Rightarrow q) \Leftrightarrow (\forall x : S \cdot p) \Rightarrow (\exists x : S \cdot q)\)

(b) \((\forall x : S \cdot p \Rightarrow q) \Rightarrow ((\forall x : S \cdot p) \Rightarrow (\forall x : S \cdot q))\)

(c) \((\exists x : S \cdot p) \Rightarrow (\exists x : S \cdot q)) \Rightarrow (\exists x : S \cdot p \Rightarrow q)\)

(d) \((\forall x : S \cdot p \Rightarrow N) \Rightarrow (\exists x : S \cdot p) \Rightarrow N\)

(e) \((\exists x : S \cdot N \Rightarrow p) \Rightarrow N \Rightarrow (\exists x : S \cdot p)\)

(f) \((\exists x : S \cdot p \Rightarrow N) \Rightarrow (\forall x : S \cdot p) \Rightarrow N\)