Boole's mathematical theory of logic and probability

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Boole's *Laws of Thought*

- I will be discussing Boole's theory as described in his 1854 book: *An Investigation of the Laws of Thought on which are founded the Mathematical Theories of Logic and Probabilities* [Boo54]

Bertrand Russell on Boole's *Laws of thought*

Pure mathematics was discovered by Boole, in a work which he called the Laws of Thought (1854). This work abounds in asseverations that it is not mathematical, the fact being that Boole was too modest to suppose his book the first ever written on mathematics. He was also mistaken in supposing that he was dealing with the laws of thought: the question how people actually think was quite irrelevant to him, and if his book had really contained the laws of thought, it was curious that no one should ever have thought in such a way before. His book was in fact concerned with formal logic, and this is the same thing as mathematics. [Rus14]

No metaphysics!

. . . we are, for all really scientific ends, unconcerned with the truth or falsehood of any metaphysical speculation whatever [Boo54, p.41]

What do atomic symbols represent in the *Investigation*?

- In modern propositional logic, atomic symbols always represent *propositions*.
- In Boole's *Primary Propositions* he has them represent things:

  1st. Literal symbols, as x, y, &c., representing things as subjects of our conceptions [Boo54, p. 27]

What sort of things are represented in Boole's logic?

> *We are permitted, therefore, to employ the symbols x, y, z, &c., in the place of the substantives, adjective, and descriptive phrases subject to the rule of interpretation, that any expression in which several of these symbols are written together shall represent all the objects or individuals to which their several meanings are together applicable, . . . [Boo54, p. 29-30]*

> *if x alone stands for "white things", and y for "sheep," let xy stand for "white sheep" [Boo54, p. 28].*

- Boole's symbols represents *sets*.
- He calls them *classes*.

Primary Propositions in the *The Laws of Thought*

- Let s be the set of all swans and w be the set of all white things; where both s and w are subsets of some "universe of discourse" [Boo54, p. 42]
- Then $s \subseteq w$ is the proposition that all swans are white

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- Then $s \subseteq w$ is the proposition that all swans are white
- $\forall x : swan(x) \rightarrow white(x)$
- Boole would actually write $s \subseteq w$ as $s = vw$ where v is "a class indefinite in every respect" [Boo54, p. 61]

Boole's logical set theory

- 0 denotes the empty set
- 1 denotes the 'universe'
- $x + y$ indicates *disjoint* union.

The expression $x + y$ seems indeed uninterpretable, unless it be assumed that the things represented by x and the things represented by y are entirely separate; that they embrace no individuals in common.

it is not true that in Logic $x + x = x$, though it is true that $x + x = 0$ is equivalent to $x = 0$ (reply to Jevons [Jou14], cited in [Bur14]).

Logic and Number

... the symbols of Logic are subject to the special law,

$$x^2 = x$$

... the equation $x^2 = x$, considered as algebraic, has no other roots than 0 and 1 ... Let us conceive, then, of an Algebra in which the symbols x, y, z, &c. admit indifferently of the values 0 and 1, and of these values alone. [Boo54, p.37]

The Rule of 0 and 1

- Burris identifies what he (not Boole) calls the *Rule of 0 and 1* as a key argument of Boole's. This rule states that:

  a law or argument held in logic iff after being translated into equational form it held in common algebra with this 0,1-restriction on the possible interpretations (i.e., values) of the symbols. [Bur14]

Relations in Boole's logic

- To express and reason about relations between objects we now use first-order logic: $\mathrm{conquered}(\mathrm{Cæsar}, \mathrm{the\_Gauls})$
- In his section on *Signs by which relation is expressed ...* Boole argues for expressing all relations using "is".

> *"Cæsar conquered the Gauls" may be resolved into "Cæsar is he who conquered the Gauls" [Boo54, p.35]*

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- $x =$ "Cæsar" and $y =$ "One who conquered the Gauls" are the same (singleton) *set*: $x = y$.
- In modern first-order logic we have predicate symbols and terms (denoting sets and individuals). Boole's logic never denotes individuals, but can use sets of which an individual is the only member.

Secondary Propositions

Logic is conversant with two kinds of relations,—relations among things, and relations among facts [Boo54, p.7]

- ‘Secondary Propositions’ “concern[], or relate[] to, other propositions regarded as true of false” [Boo54, p.159]
- Secondary Propositions are “subject to the same laws of combination as the corresponding symbols employed in the expression of Primary Propositions” [Boo54, p.165].

Secondary Propositions and time

x denotes the time for which the proposition X is true [Boo54, p.165].

- x is the *representative symbol* for X.
- As a Secondary Proposition 0 means never (empty set of "successive moments") and 1 means always (contains all moments).
- The Secondary Proposition $x = y$ does not mean that x and y "are identical, but that the times of their occurrence are identical". [Boo54, p.176]

Boole's probability

. . . . . . the subject of Probabilities belongs equally to the science of Number and to that of Logic. In recognising the co-ordinate existence of both these elements, the present treatise differs from all previous ones [Boo54, p. 13]

Using logic to compute probabilities

. . . there exists a definite relation between the laws by which probabilities of events are expressed as algebraic functions of the probabilities of other events upon which they depend, and the laws by which the logical connexion of the events is itself expressed [Boo54, p. 16]

Independence from ignorance

The events whose probabilities are given are to be regarded as independent of any connexion but such as is either expressed, or necessarily implied, in the data; and the mode in which our knowledge of that connexion is to be employed is independent of the nature of the source from which such knowledge has been derived. [Boo54, p. 256-7]

The simple events, x, y, z, will be said to be "conditioned" when they are not free to occur in every possible combination; in other words, when some compound event depending upon them is precluded from occurring. . . . Simple unconditioned events are by definition independent. [Boo54, p. 256-7]

Primary combinations

EVENTS.		PROBABILITIES.
xy	Concurrence of x and y	pq.
$x(1-y)$	Occurrence of x without y	$p(1-q)$.
$(1-x)y$	Occurrence of y without x	$(1-p)q$.
$(1-x)(1-y)$	Conjoint failure of x and y	$(1-p)(1-q)$.

[Boo54, p. 259]

Independence from ignorance justified

To meet a possible objection, I here remark, that the above reasoning does not require that the drawings of a white and a marble ball should be independent, in virtue of the physical constitution of the balls. The assumption of their independence is indeed involved in the solution, but it does not rest upon any prior assumption as to the nature of the balls, and their relations, or freedom from relations, of form, colour, structure, &c. It is founded upon our total ignorance of all these things. Probability always has reference to the state of our actual knowledge, and its numerical value varies with varying information. [Boo54, p. 262-3]

Keynes' objection

The central error in his system of probability arises out of his giving two inconsistent definitions of 'independence' (2) He first wins the reader's acquiescence by giving a perfectly correct definition: "Two events are said to be independent when the probability of either of them is unaffected by our expectation of the occurrence or failure of the other." (3) But a moment later he interprets the term in quite a different sense; for, according to Boole's second definition, we must regard the events as independent unless we are told either that they must concur or that they cannot concur. . . . In fact as long as xz is possible, x and z are independent. [Key21]

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- Keynes exaggerates Boole's position.
- x and y are independent iff none of xy, $(1-x)y$, $(1-x)(1-y)$,$(1-x)(1-y)$ are ruled out.
- Although it is true that we could have none of these ruled out and still have our expectation of x affected by the occurrence of y
- But perhaps that would be because x and y are not 'simple'.

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- But perhaps that would be because x and y are not 'simple'.
- Note that Wilbraham pointed out Boole's alleged contradiction in a review (also written in 1854) of *Laws of Thought*. Boole did not understand Wilbraham's criticism and "replied hotly, challenging him to impugn any individual results" [Key21, p. 167]

Computing probability intervals

- In some cases, not all the required probability values are given in which case the desired probability value cannot be returned, only the interval in which it lies. Deductive relations provide constraints between probabilities.

Was Boole's method of computing probabilities valid?

Boole's own method of solving [some problems] is constantly erroneous, and the difficulty of his method is so great that I do not know of anyone but himself who has ever attempted to use it. [Key21] (quoted in [Gow])

Was Boole's method of computing probabilities valid?

Following Gow's presentation where A_1' is the complement of A_1.
Given: $p(A_1) = c_1$, $p(A_2) = c_2$, $p(E|A_1) = p_1$, $p(E|A_2) = p_2$ and that $E \cap A_1' \cap A_2' = \emptyset$.
Compute: $u = p(E)$.

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Compute: $u = p(E)$.
Boole derives this:

$$\frac{(u - c_1 p_1)(u - c_2 p_2)}{c_1 p_1 + c_2 p_2 - u} = \frac{[1 - c_1(1 - p_1) - u][1 - c_2(1 - p_2) - u]}{1 - u}$$

The only valid solution for u being (apparently)

$$u = \frac{ab - a'b' + (1 - a' - b')c' + \sqrt{Q}}{2(1 - a' - b')}$$

where $a' = 1 - c_1(1 - p_1)$, $b' = 1 - c_2(1 - p_2)$, $c' = c_1 p_1 + c_2 p_2$,
$Q = [ab - a'b' + (1 - a' - b')c']^2 - 4(1 - a' - b')(ab - a'b'c')$

Wilbraham and Gow's correct solution

Following Gow's presentation where A'_1 is the complement of A_1.
Given: $p(A_1) = c_1$, $p(A_2) = c_2$, $p(E|A_1) = p_1$, $p(E|A_2) = p_2$ and that $E \cap A'_1 \cap A'_2 = \emptyset$.
Compute: $u = p(E)$.

$$\begin{aligned} p(E) &= p(E \cap A_1 \cup E \cap A_2) \\ &= p(E \cap A_1) + p(E \cap A_2) - p(E \cap A_1 \cap A_2) \\ &= c_1 p_1 + c_2 p_2 - p(E \cap A_1 \cap A_2) \end{aligned}$$

Since $p(E \cap A_1 \cap A_2)$ is not given the solution is undetermined.

> *Wilbraham gave as the solution $u = c_1 p_1 + c_2 p_2 - z$, where z is necessarily less than either $c_1 p_1$ or $c_2 p_2$. This solution is correct so far as it goes, but is not complete. (Keynes, quoted in in [Gow])*

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Since $p(E \cap A_1 \cap A_2)$ is not given the solution is undetermined. Wilbraham worked out that Boole assumed two equations to reach his solution. Gow has confirmed Wilbraham's result

$$\frac{p(A_1 \cap A_2 \cap E)}{p(A_1' \cap A_2 \cap E)} = \frac{p(A_1 \cap A_2' \cap E)}{p(A_1' \cap A_2' \cap E')}$$

$$\frac{p(A_1 \cap A_2 \cap E')}{p(A_1' \cap A_2 \cap E')} = \frac{p(A_1' \cap A_2 \cap E')}{p(A_1' \cap A_2' \cap E')}$$

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Since $p(E \cap A_1 \cap A_2)$ is not given the solution is undetermined.
If we could assume $P(A_1 \cap A_2|E) = P(A_1|E)P(A_2|E)$ then we get two solutions: $u = c_1 p_1$, $u = c_2 p_2$.

Boole's influence

Boole's probabilistic logic is of the highest relevance today, since it provides a basis for dealing with uncertainty in knowledge-based systems that is not only well grounded theoretically but has some practical advantages as well. [AH94]

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The fundamental problem of probabilistic inference is to determine the probability of a conclusion that is inferred from uncertain premises . . . T. Hailperin pointed out that this problem can be naturally captured in a linear programming model, which Boole himself all but formulated. About a decade later N. Nilsson reinvented probabilistic logic and its linear programming formulation [], and his paper sparked considerable interest in the artificial intelligence community [AH94]

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