Handling first-order linear constraints with SCIP

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Mixed integer programs

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Variables
These can be binary, integer or real-valued

Constraints
These are linear inequalities (maybe equations)

Objective
\[ z = c_i x_i \]
where each variable \( x_i \) has an objective coefficient \( c_i \).

▶ Goal is to assign values to the variables which meet all constraints and maximise the objective.

▶ If all variables are real-valued we have a linear program (LP) which can be solved in polynomial time.

▶ Otherwise the linear relaxation provides a useful upper bound.
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- Goal is to assign values to the variables which meet all constraints and maximise the objective.
- If all variables are real-valued we have a *linear program (LP)* which can be solved in polynomial time.
- Otherwise the *linear relaxation* provides a useful upper bound.
- Can extend to *constraint integer program* (SCIP approach).
Facility location in ZIMPL

set STORES := { 1 .. 9 };
set PS := PLANTS * STORES;
var x[PS] binary; # Is plant p supplying store s?
var z[PLANTS] binary; # Is plant p build?
minimize cost: sum <p> in PLANTS : building[p] * z[p]
+ sum <p,s> in PS : transport[p,s] * x[p,s];
subto assign: forall <s> in STORES :
    sum <p> in PLANTS : x[p,s] == 1;
subto build: forall <p,s> in PS : x[p,s] <= z[p];
subto limit: forall <p> in PLANTS :
    sum <s> in STORES : demand[s] * x[p,s] <= capacity[p];
First-order representations for convenience

► A good modelling language makes it easy to define large mixed integer programs.
► Here’s how to define a Markov logic network in the modelling language ZIMPL.
► Show univ.mln and univ.zpl
Grounding the first-order representation

- The ZIMPL description can be translated to a standard MIP format (e.g. .lp),
- or sent directly to SCIP.
- In either case the compact representation is grounded before solving begins
- Show univ.lp
- Demo solving of univ.zpl
What’s wrong with grounding?

- If there are not ‘too many’ variables and constraints in the ground MIP then grounding is a good option.
- But what to do if there are too many?
The obvious solution

- Compact MIPs are often reformulated (typically by a human) to have very many linear constraints and/or very many variables.
- This is to end up with a MIP whose LP relaxation is ‘tight’ (possibly even integer).
- The (typically correct) assumption is that most constraints are not tight and most variables have value zero.
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- This is to end up with a MIP whose LP relaxation is ‘tight’ (possibly even integer).
- The (typically correct) assumption is that most constraints are not tight and most variables have value zero.
- Most linear constraints are added only when/if useful during the course of solving (*cutting planes*).
- Most variables are created only when/if useful during the course of solving (*column generation*).
- Not-yet-existing variables implicitly have their value clamped to zero.
The obvious solution

- Instead of grounding a first-order linear constraint prior to solving, useful ground instances can be generated as cutting planes.
- Instead of creating all variables prior to solving, useful variables can be created using column generation.
First steps

- I have an initial implementation of this approach called foilp.
- It accepts input in a language (‘FOZ’) virtually identical to ZIMPL.
- Show smoke.zpl and smoke.foz
- A Python script converts the FOZ file to a ZIMPL file with special fake variables and constraints (Show smoke_fo.zpl).
- foilp uses that information to create first-order clausal constraints.
- Show solving with smoke.zpl and smoke_fo.zpl.
Using SCIP

- foilp is just SCIP extended with two new constraint handlers: foclause and folinear.
- A SCIP constraint handler must do (at least) the following:
  - **CONSCHECK** Decide whether an arbitrary candidate solution satisfies all the constraints (of the constraint handler).
  - **CONSENFOLP** Decide whether an LP solution satisfies all the constraints (of the constraint handler).
  - **CONSENFOPS** Decide whether a ‘pseudo-solution’ satisfies all the constraints (of the constraint handler).
  - **CONSLOCK** Specify whether the constraints lock variables up or down (or both).
When to add cutting planes

- foilp implements the CONSSEPALP method to generate cutting planes
- Recall that at the beginning no ground instances of a first-order constraint have been posted.
- So we have an initial under-constrained problem which may still be tough to solve if there are other constraints.
- We do not wait until this problem is solved before adding ground constraints—this could take a while.
- We add ground instances (cutting planes) as soon as the LP relaxation of the under-constrained problem is solved.
- The LP relaxation removes the constraint that variables have to take integer values, and can be solved quickly (polynomial time).
- Once the cutting planes are added we solve the new LP and so on . . .
Recall that literals are binary variables in the MIP which may have fractional values in the LP solution.
Suppose the LP solution includes the following values:
\( p(a) = 0.2, q(a, b) = 0.7, r(b) = 0. \)

And our problem has the following first-order clause:
\[
\forall x, y : p(x) \lor \neg q(x, y) \lor r(y)
\leq \forall x, y : p(x) + 1 - q(x, y) + r(y) \geq 1
\]

Then \( p(a) + 1 - q(a, b) + r(b) \geq 1 \)
is a cutting plane worth adding—it *separates* the current LP solution, since
\( 0.2 + 1 - 0.7 + 0 < 1 \)
Efficient (?) separating

- In general deciding whether a first-order clause is satisfied by an interpretation is NP-complete (since we can encode any graph colouring problem like this).
- If the interpretation may be fractional then it will not get any easier!
- Current implementation of foilp does the sensible thing on clauses where the literals do not share variables.
- Otherwise just enumerates ground instances — it would not be hard to do better!
Proposal: Use a sub-IP to find a cutting plane

# Each first-order variable has exactly one grounding
\[ I(x=a) + I(x=b) + \ldots + I(x=c) = 1; \]
# Groundings of literals must be
# consistent with that of variables
\[ I(x=a) = I(p(a,a)) + I(p(a,b)) + \ldots + I(p(a,c)); \]
# Want grounding to violate first-order clause
\[
\begin{align*}
\text{minimize } \sum <x,y>: \ & \text{LPval}(p[x])*I(p[x]) \\
& + 1-\text{LPval}(q[x,y])*I(q[x,y]) \\
& + \text{LPval}(r[y])*I(r[y]) \\
\end{align*}
\]

Solution must be below 1 for there to be an associated cutting plane
What about variable creation?

There are two reasons to create a new variable:

1. If there is no feasible solution to the LP relaxation.
2. If there is a better solution where the new variable has a positive value.
Creating variables (column generation)

- Here we follow the presentation given by Gamrath [Gam10].
- Let the current LP be “min $cx$ subject to $Ax \geq a$”, where $x$ is the vector of currently-existing variables. Suppose there is a feasible solution to this LP.
- The LP solver will provide us with $\pi^*$, the optimal solution to the dual of this LP. Each component of $\pi^*$ corresponds to a row of $A$ (a linear constraint).
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Let $c_i$ be the objective coefficient for a potential new variable $x_i$.

Let $A_{.,i}$ be the vector of its coefficients in the current set of linear constraints.

The *pricing problem* is to find a new variable $x_i$ with minimal reduced cost:

$$c^* = \min \{ c_i - A_{.,i}\pi^* \}$$
Recovering from infeasibility

If we have the clausal constraint:

\[ p(x) + q(x, y) \geq 1 \]

and, say, the MIP binary variables \( p(a) \) and \( p(a, b) \) do not exist yet then the constraint handler may generate the following cutting plane:

\[ 0 \geq 1 \]

We need to create \( p(a) \) or \( p(a, b) \).

- If the LP is infeasible, the LP solver will provide Farkas multipliers \( u^T \).
- Pricing problem is then \( c^* = \min\{-A_{i:,}^T u\} \).
- In practice typically better to create variables to ensure feasibility at the beginning.
Future work, etc

The most successful approach to date is to ensure that only one isomorphic copy of each node is kept in the enumeration tree [CCZ14].

Generate cuts using resolution as done by Chandru and Hooker [CH99].

Result: derived clauses can only be facets if they are prime implicants.

Generate new (useful!) first-order constraints from existing ones using first-order resolution.
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