Symmetry
and the
Generation of Constraint Models

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Constraint Programming

Constraint Programming (CP) is useful for solving a wide range of important, complex problems
– scheduling, allocation, layout, configuration, …

Def (Narrow view): Solve combinatorial (optimisation) problem by

• **Model** it by mapping it to a finite-domain constraint satisfaction problem
• **Solve** the constraint satisfaction problem
• **Map** the solution back to original problem
Symmetry

- Models often contain an enormous number of symmetries.
- Symmetry in a model yields symmetry in the space that is searched for a solution, which leads to inefficiency.
- Once symmetries in the model are identified, there are a variety of methods for removing it from the model or from the search.
- **How do we identify them?**
Conclusion

• Symmetries enter a model from two sources:
  – inherent in the problem
  – introduced by the modelling process
• The modelling process can be formalised and automated [IJCAI-05].
• A formal/automated account of modelling should provide a formal/automated account of the symmetries introduced by modelling.
The Plan

- Constraint satisfaction problems and their symmetries
- An introduction to modelling (by example)
- The automation of modelling
- Automatic identification of symmetry introduced by modelling
Part 1

Constraint Satisfaction Problems and their Symmetries
What is the (finite domain) CSP?

An instance comprises:
• Finite set of variables
• Each associated with a finite domain
• Finite set of constraints on the values taken by the variables
• (Objective function)

Solution is an assignment of values to variables that satisfies the constraints (and optimises objective function)
Example of a CSP

Problem: Find three distinct digits that sum to 4.

\[ X, Y, Z \text{ in } \{0,1,2,3,4,5,6,7,8,9\} \]
\[ X \neq Y, \ Y \neq Z, \ X \neq Z \]
\[ X + Y + Z = 4 \]

A Solution: \( \{X \rightarrow 0, \ Y \rightarrow 1, \ Z \rightarrow 3\} \)
What Is Symmetry?

- A symmetry is a total bijective function from total instantiations to total instantiations which maps every solution to a solution.
- Usually consider a set of symmetries that is closed under inverse and composition and to contain identity function, i.e. a group.
- This induces a partitioning on the total instantiations. Each partition is called a symmetry class. Every member is a solution or no member is.
- In general, other objects can be mapped and other properties preserved.
Value Symmetry

• A group of symmetries on values that induces a group of symmetries on instantiations.

• Example: “Assign r, b or g to each node of a graph so that no arc connects two nodes of the same colour” has value symmetry. 
  \( S = \{ r \rightarrow b, b \rightarrow g, g \rightarrow r \} \).

• Induces a symmetry on instantiations that maps 
  \( \{ N1 \rightarrow r, N2 \rightarrow b, N3 \rightarrow g \} \) to 
  \( \{ N1 \rightarrow S(r), N2 \rightarrow S(b), N3 \rightarrow S(g) \} = \{ N1 \rightarrow b, N2 \rightarrow g, N3 \rightarrow r \} \).
Variable Symmetry

• A group of symmetries on variables that induces a group of symmetries on instantiations.

• Example: “Assign 0,…,9 to X, Y, Z such that X, Y, Z are all different and sum to 4” has variable symmetry.
  \[ S = \{X \rightarrow Y, Y \rightarrow Z, Z \rightarrow X\} \].

• Induces a symmetry on instantiations that maps
  \[ \{X \rightarrow 0, Y \rightarrow 1, Z \rightarrow 3\} \] to
  \[ \{S(X) \rightarrow 0, S(Y) \rightarrow 1, S(Z) \rightarrow 2\} = \{Y \rightarrow 0, Z \rightarrow 1, X \rightarrow 2\} \].
Partial Instantiation Search
(Forward Checking)
How to Break Symmetries

• Be clever during search (SBDS, SBDD, …)
• Reformulate the problem
• Add symmetry breaking constraints to problem formulation.
  – Consistent: At least one instantiation in every symmetry class satisfies the constraints.
  – Complete: At most one instantiation in every symmetry class satisfies the constraints.
  – Enforcing these constraints during search prunes paths---solution paths and failure paths.
Part 2

An Introduction to Modelling
(by example)
Model using Explicit Representation

Given \( n \) and \( s \),
Find a set of \( n \) digits that sum to \( s \).

\[
\begin{array}{cccccc}
1 & 2 & 3 & \cdots & n \\
0..9 & 0..9 & 0..9 & \cdots & 0..9 \\
\end{array}
\]

given \( n \): int, \( s \): int
find \( X \): matrix (indexed by 1..\( n \)) of int (0..9)
such that \( \text{AllDiff}(X) \)
\( \text{Sum}(X) = s \)

Symmetries: permutations of the index values
Model using Occurrence Representation

Given \( n \) and \( s \),

Find a set of \( n \) digits that sum to \( s \).

\[
\begin{array}{cccccc}
0 & 1 & 2 & \ldots & 9 \\
0/1 & 0/1 & 0/1 & \ldots & 0/1 \\
\end{array}
\]

**given** \( n \): int, \( s \): int  
**find** \( X \): matrix (indexed by 0..9) of int (0..1)  
**such that** \( \sum_{i \in 0..9} i \cdot X[i] = s \)

**Symmetries: none**
The SONET Problem

**Specification**

- Given \( nrings \) rings, \( nnodes \) nodes, a set of pairs of nodes (communication demand) and an integer capacity (of each ring). Install nodes on rings satisfying demand and capacity constraints. Minimise installations.

**Instance**

- \( nrings = 2 \), \( nnodes = 5 \), \( capacity = 4 \)

- Demand: \( n_1 \& n_3, \ n_1 \& n_4, \ n_2 \& n_3, \ n_2 \& n_4, \ n_3 \& n_5 \)

**Solution**

![Diagram showing the solution with nodes and rings]
Model of the SONET Problem

<table>
<thead>
<tr>
<th>Nodes</th>
<th>Rings</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>0.1</td>
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<tr>
<td>0.1</td>
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</tr>
</tbody>
</table>

**Symmetries:**
- Permute the 2nd index (columns)
The Social Golfers Problem

A number of golfers play a round of golf in each of $n$ weeks. Each week they are divided into $ngroups$, groups, each of which plays together. To maximise socialisation, no two golfers can play together twice. How should the golfers be divided each week?

<table>
<thead>
<tr>
<th>weeks</th>
<th>groups</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[1,2,3]</td>
</tr>
<tr>
<td></td>
<td>[4,5,6]</td>
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<tr>
<td></td>
<td>[7,8,9]</td>
</tr>
<tr>
<td></td>
<td>[1,4,7]</td>
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<tr>
<td></td>
<td>[2,5,8]</td>
</tr>
<tr>
<td></td>
<td>[3,6,9]</td>
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<tr>
<td></td>
<td>[1,5,9]</td>
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<tr>
<td></td>
<td>[2,6,7]</td>
</tr>
<tr>
<td></td>
<td>[3,4,8]</td>
</tr>
</tbody>
</table>
**Model Social Golfers Problem**

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<td>[1,2,3]</td>
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<td>[2,5,8]</td>
</tr>
<tr>
<td>[1,5,9]</td>
<td>[2,6,7]</td>
</tr>
<tr>
<td></td>
<td>[3,4,8]</td>
</tr>
</tbody>
</table>

**Symmetries**

- Permutations of the values
- Permutations of the first index (rows)
- Permutations of the second index (columns)
- Permutation of the third index
Model Social Golfers Problem

<table>
<thead>
<tr>
<th>weeks</th>
<th>groups</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[1,2,3]</td>
</tr>
<tr>
<td></td>
<td>[4,5,6]</td>
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<td>[1,5,9]</td>
<td>[3,4,8]</td>
</tr>
</tbody>
</table>

Symmetries

- Permutations of the values
- Permutations of the first index (rows)
- Within each row, permutations of the 2nd index (cols)
- Within each cell, permutations of the 3rd index
Part 3

Automated Modelling
The Modelling Bottleneck

• Modelling a problem as a constraint program requires moderate/great expertise.
  – The model chosen has a substantial effect on efficiency of solving

• Major barrier to widespread use of Constraint Programming.
Reducing the Modelling Bottleneck

• Systematise the knowledge of the expert.
  – ~5 years: We have been building models more and more systematically

• Embed this knowledge in a “compiler” that can refine a high-level problem specification into a set of constraints that can be executed efficiently using existing toolkits.
CONJURE
Automated Model Generation

• Given an abstract formal problem specification, in ESSENCE

• Generates a set of correct models
  – Should include good models generated by experts
  – These models are in ESSENCE', a subset of ESSENCE similar to what is provided by existing constraint toolkits
  – ESSENCE must enable specification at a level of abstraction above that at which modelling decisions are made
ESSENCE

• The language enables problems to be specified at a level of abstraction above that at which modelling decisions are made.
  – Allow problems to be specified without making modelling decisions

• This requires features not found in current constraint programming languages.
SONET Specification

given \( n \): int, \( m \): int, \( c \): int

demand: set of set (size 2) of int (1..\( m \))

find \( \text{rings} \): mset (size \( n \)) of set (maxsize \( c \)) of int (1..\( m \))

minimising \( \sum_{r \in \text{rings}} |r| \)

such that \( \forall \text{pair} \in \text{demand}. \exists r \in \text{rings}. \text{pair} \subseteq r \)
ESSENCE Provides Abstract Types

Sets, Multisets, Partitions, Unnamed Types
Functions, Relations, Tuples

given $n \cdot \text{int}, m \cdot \text{int}, c \cdot \text{int}$

demand: set of set (size 2) of int (1..$m$)

find $rings: \text{mset} (\text{size } n) \text{ of set} (\text{maxsize } c) \text{ of int (1..$m$})$

minimising $\sum_{r \in rings} |r|$ \( \leq \)

such that $\forall \text{pair} \in \text{demand}. \exists r \in rings . \text{pair} \subseteq r$
ESSENCE Supports Arbitrarily-Nested Types

given
\[ n: \text{int}, \ m: \text{int}, \ c: \text{int} \]
\[ \text{demand}: \ \text{set of set (size 2) of int (1..m)} \]

find
\[ \text{rings}: \ \text{mset (size n) of set (maxsize c) of int (1..m)} \]

minimising
\[ \sum_{r \in \text{rings}} |r| \]

such that
\[ \forall \text{pair} \in \text{demand}. \ \exists r \in \text{rings}. \ \text{pair} \subseteq r \]
ESSENCE Supports Quantification over Decision Variables

given \( n: \text{int}, m: \text{int}, c: \text{int} \)

\( \text{demand}: \text{set of set (size 2) of int (1..m)} \)

find \( \text{rings}: \text{mset (size } n \text{) of set (maxsize } c \text{) of int (1..m)} \)

minimising \( \sum_{r \in \text{rings}} |r| \)

such that \( \forall \text{pair} \in \text{demand}. \exists r \in \text{rings}. \text{pair} \subseteq r \)
How Usable is ESSENCE?

• Specifications of ~50 problems found in the CSP literature written by an undergraduate with no background in constraint programming.

• URL: http://www.cs.york.ac.uk/aig/constraints/
ESSENCE’ has a level of abstraction similar to existing constraint languages (OPL, Solver, ECLiPSe, …)

ESSENCE’ = ESSENCE -

- Abstract types
- Arbitrary nesting of types
- Quantifying Over Decision Variables
Formalisation of the Modelling Problem

**ESSENCE**

- **given**: \( n: \text{int}, s: \text{int} \)
- **find**: \( X: \text{set (size } n\text{)} \text{ of int}(0..9) \)
- **such that**: \( \sum_{x \in X} x = s \)

**ESSENCE’**

- **given**: \( n: \text{int}, s: \text{int} \)
- **find**: \( X: \text{matrix (indexed by 1..n)} \text{ of int}(0..9) \)
- **such that**: \( \text{AllDiff}(X) \)
  \[ \text{Sum}(X) = s \]

**ESSENCE’’**

- **given**: \( n: \text{int}, s: \text{int} \)
- **find**: \( X: \text{matrix (indexed by 0..9)} \text{ of int}(0..1) \)
- **such that**: \( \sum_{i \in 0..9} i \cdot X[i] = s \)
  \[ \text{sum}(X) = n \]
Part 4

Automated Identification of Symmetries Introduced by Modelling
CONJURE Identifies Symmetries

- CONJURE annotates the models it produces with the symmetries it introduces.
- As a model is generated, each refinement rule generates appropriate annotations.
## Refinement of the SONET Problem

**Refine:** rings:mset (size nrings) of set (size capacity) of Nodes

<table>
<thead>
<tr>
<th>Refine</th>
<th>mset (size n) of ( \tau )</th>
<th><strong>Explicit</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>To</td>
<td>a matrix (indexed by 1..n) of refine(( \tau ))</td>
<td></td>
</tr>
<tr>
<td>Sym</td>
<td>permutations of the index values of the matrix symmetries introduced by refine(( \tau ))</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Refine</th>
<th>set (size n) of ( \tau )</th>
<th><strong>Occurrence</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>To</td>
<td>a matrix (indexed by ( \tau )) of bool</td>
<td>none</td>
</tr>
</tbody>
</table>

*provided \( \tau \) is bool, finite set of int, enumerated type*
Model of the SONET Problem

<table>
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<tr>
<td>0..1</td>
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Symmetries:
- Permute the 2nd index (columns)
### Refinement of the Golfers Problem

**Refine:** schedule: mset (size nweeks) of regpart (size ngroups) of golfers

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<table>
<thead>
<tr>
<th>Refine</th>
<th>\textbf{mset (size n) of }\tau \textbf{ Explicit}</th>
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<tbody>
<tr>
<td>To</td>
<td>a matrix (indexed by 1..n) of refine(\tau)</td>
</tr>
<tr>
<td>Sym</td>
<td>permutations of the index values of the matrix</td>
</tr>
<tr>
<td></td>
<td>each element of the matrix independently has</td>
</tr>
<tr>
<td></td>
<td>every symmetry of refine(\tau)</td>
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<thead>
<tr>
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<th>\textbf{regpart (size n) of }\tau</th>
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<tr>
<td>To</td>
<td>a matrix (indexed by 1..n, 1..</td>
</tr>
<tr>
<td>Sym</td>
<td>permutations of the values of the 1\textsuperscript{st} index</td>
</tr>
<tr>
<td></td>
<td>within each row, permutations of values of 2\textsuperscript{nd} index</td>
</tr>
</tbody>
</table>

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<table>
<thead>
<tr>
<th>Refine</th>
<th>\textbf{newtype (size n)}</th>
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<tbody>
<tr>
<td>To</td>
<td>1..n</td>
</tr>
<tr>
<td>Sym</td>
<td>representations of the new type</td>
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Further Information

- http://www.cs.york.ac.uk/aig/constraints/
  - Our papers
  - Catalogue of CONJURE rules
  - Syntax and semantics of ESSENCE version 1
  - Catalogue of ~50 problems specified in ESSENCE and other constraint languages