

# Computation of Noise Spectral Density in Switched Capacitor Circuits using the Mixed-Frequency-Time Technique

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## ABSTRACT

A time-domain algorithm for computation of the noise power spectral density (PSD) is proposed in [17]. When applied to periodically varying circuits, this formulation requires efficient methods to obtain the quasi-periodic steady-state solution of a set of linear time-varying ordinary differential equations. We use both the mixed frequency-time technique(MFT) and the shooting Newton method for this computation. We show that, at each frequency for which the spectral density is sought, MFT requires only two integrations over a duration of the clock period. The results are compared with published experimental and computed data.

## Categories and Subject Descriptors

J.2 [Physical Sciences and Engineering]: Electronics;  
B.7.2 [Design aids]: Simulation

## General Terms

Algorithms

## Keywords

Noise computation, switched capacitor circuits

## 1. INTRODUCTION

Computation of noise spectral density in switched capacitor circuits has received considerable attention. Several algorithms have been proposed [2, 5, 7, 11, 13, 16, 19]. All these are essentially frequency domain methods. It would be advantageous to have an efficient time domain technique that can easily be integrated into a standard circuit simulator such as SPICE. A step in this direction was taken by Demir *et al* [1]. They obtained a set of ordinary differential equations for the time-varying covariance matrix. In [17],

a time-domain method to obtain the average noise power spectral density is proposed. It is based on the fact that the power spectral density is the limiting value of the expected energy spectral density per unit time.

When applied to periodically varying circuits, this formulation requires efficient techniques for computation of the quasi-periodic steady state solution of a set of linear time-varying ordinary differential equations. Other than direct integration, two methods that are commonly used to get steady state solutions are the shooting Newton technique [15] and the mixed frequency/time method(MFT) [3, 8, 9, 10]. The shooting Newton technique is applicable to circuits with a periodic steady state response. In this method, transient simulations have to be performed for at least one period of the output signal to get the corrected initial state. This has some disadvantages in the case of switched capacitor circuits, where the clock frequency is usually much greater than the signal bandwidth. A single output period could very easily include a few hundred clock periods. The output period also depends on the relationship between the frequency at which the PSD is desired,  $\omega_m$ , and the clock frequency,  $\omega_c$ . Therefore, the computation of the noise spectrum using this method could mean long transient simulations. The MFT method can be used to circumvent some of these difficulties. In this method, the transient solution, computed over a few selected clock cycles, can be used to construct the quasi-periodic output signal accurately. This method has been demonstrated to be very efficient for computation of the steady state output signal as well as distortion in large switched capacitor filter circuits [3, 6].

In this paper, we use MFT along with the method proposed in [17] to obtain the noise spectral density in switched capacitor circuits. For each value of  $\omega_m$ , we show that the method requires only two integrations over a clock cycle. Moreover, due to the nature of the differential equations involved, the MFT method can be used to obtain the noise spectrum for all frequencies, including the clock frequency and its harmonics. At the clock frequency and its harmonics, it degenerates to the shooting Newton method.

This paper is organized as follows. Section 2 contains the algorithm for computation of the noise spectral density. In section 3, the application of the MFT method to noise spectral density computation is discussed. Section 4 contains the results of the computations performed

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## 2. ALGORITHM FOR COMPUTATION OF THE NOISE SPECTRAL DENSITY

As with all other techniques, noise is treated as a perturbation. Only thermal noise sources are considered. This includes noise due to the switches and opamps. In this paper, we use the state variable form of the equations. However, the modified nodal equations can also be used.

The state variable form of the circuit equations without noise sources can be written as:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}(\mathbf{x}(t), \mathbf{v}(t)) \quad (1)$$

where  $\mathbf{x}(t)$  is the vector containing the state variables of the circuit and  $\mathbf{v}(t)$  is the vector of large signal excitations to the circuit. This can be solved to get the large signal steady state solution of the circuit,  $\mathbf{x}_s(t)$ . With the noise sources added, the solution is assumed to be of the form:

$$\mathbf{x}(t) = \mathbf{x}_s(t) + \mathbf{x}_n(t)$$

where  $\mathbf{x}_n(t)$  represents the noise voltages. Since noise is treated as a perturbation, a linearized form of the state equations along with additive noise sources can be used for noise computations. The large signal input sources are set to zero. The linearized equations can be written as:

$$\frac{d\mathbf{x}_n(t)}{dt} = \mathbf{A}(t)\mathbf{x}_n(t) + \mathbf{B}(t)\mathbf{u}(t) \quad (2)$$

Here,  $\mathbf{A}(t)$  is the Jacobian of  $\mathbf{F}(\cdot)$  computed at steady state.  $\mathbf{B}(t)$  is a matrix containing the spectral intensity of the noise sources and  $\mathbf{u}(t)$  is the vector containing the various noise sources, all of which are assumed to be standard Gaussian white noise processes uncorrelated with each other.

Assume that the noise spectral density is to be determined at node 'N' of the circuit. Let  $x_{nN}(t)$  be the noise waveform at this node. For convenience, it is assumed that  $x_{nN}(t)$  is also a state variable. We define:

$$X(t, \omega) = \int_0^t x_{nN}(\tau) e^{-j\omega\tau} d\tau \quad (3)$$

$X(t, \omega)$  is essentially the Fourier transform of a "t-segment" of the noise waveform. It is then possible to define the expected energy spectral density of this finite segment of the waveform,  $\mathcal{E}(t, \omega)$  as:

$$\mathcal{E}(t, \omega) = E\{|X(t, \omega)|^2\}$$

Here  $E\{\cdot\}$  denotes the expectation operator. The expected power spectral density is given by [4, 18]:

$$PSD = \lim_{t \rightarrow \infty} \frac{\mathcal{E}(t, \omega)}{t} \quad (4)$$

Using the methodology of stochastic differential equations, it can be shown that the energy spectral density and the cross-spectral density,  $\mathbf{K}'(t)$ , are given as [17]:

$$\frac{d\mathcal{E}(t, \omega)}{dt} = E\{x_{nN}(t)X(t, \omega)^*\}e^{-j\omega t} + E\{x_{nN}^*(t)X(t, \omega)\}e^{j\omega t} \quad (5)$$

$$\frac{d\mathbf{K}'(t)}{dt} = \mathbf{A}(t)\mathbf{K}'(t) + E\{\mathbf{x}_n(t)x_{nN}^*(t)\}e^{j\omega t} \quad (6)$$

where  $\mathbf{K}'(t)$  is a vector with:

$$K'_i(t) = E\{x_{ni}(t)X(t, \omega)^*\}$$

The second term in equation (6) is a vector containing the time-varying variance at the output node and its cross correlation with other nodes in the circuit. This can be obtained by solving the differential equations for the time-varying covariance matrix, given as [1, 14]:

$$\frac{d\mathbf{K}(t)}{dt} = \mathbf{K}(t)\mathbf{A}(t)^*T + \mathbf{A}(t)\mathbf{K}(t) + \mathbf{B}(t)\mathbf{B}(t)^T \quad (7)$$

where

$$K_{ij}(t) = E\{x_{ni}(t)x_{nj}^*(t)\}$$

A more detailed derivation of the equations and its application to switched capacitor circuits can be found in [17].

Equations (5), (6) and (7) have to be solved in order to get the output noise spectral density. In the next section, efficient techniques to perform this computation are discussed.

## 3. APPLICATION OF THE MFT METHOD

Since noise is regarded as a perturbation, equations (6) and (7) are linear time-varying equations. Moreover, since both the input vector [consisting of the noise sources] and the state matrix vary periodically with the clock, the steady state covariance matrix is a periodic function of the clock. The shooting Newton method can therefore be used very efficiently to get the periodic steady state covariance matrix. The equations for the cross-spectral densities, however, contain both the "measurement" frequency,  $\omega_m$  and the clock frequency,  $\omega_c$ . The "input" vector in (6) is effectively a product of a tone at  $\omega_m$  and the vector,  $\mathbf{K}_N(t)$ , containing the output variance and cross-correlations with other nodes. Since  $\mathbf{K}_N(t)$  is a periodic function of the clock, its elements can be expanded as a Fourier series in terms of the clock. Therefore, equation (6) can be written as:

$$\left. \frac{d\mathbf{K}'(t)}{dt} \right|_k = \mathbf{A}_k\mathbf{K}'(t) + e^{j\omega_m t} \sum_{n=-\infty}^{+\infty} \mathbf{c}_n e^{jn\omega_c t} \quad (8)$$

where  $k$  represents the  $k^{th}$  clock phase and  $\mathbf{A}_k$  is the state matrix in the  $k^{th}$  clock phase. The solution  $\mathbf{K}'(t)$  will therefore contain components at frequencies  $\omega_m \pm n\omega_c$ . Since the differential equations are linear, no other harmonics of  $\omega_m$  will be generated. Therefore,  $\mathbf{K}'(t)$  can be written as:

$$\mathbf{K}'(t) = e^{j\omega_m t} \sum_{n=-\infty}^{+\infty} \mathbf{d}_n e^{jn\omega_c t} \quad (9)$$

i.e. it contains a tone at  $\omega_m$  and several components at  $\omega_m \pm n\omega_c$ . As seen from equation (5), in order to compute the power spectral density, we require the Fourier component of the output cross-spectral density  $K'_N(t)$  at  $\omega_m$ .

Both the shooting Newton and MFT technique were used to obtain the output PSD. In the shooting Newton technique, starting with an initial guess for  $\mathbf{K}'(0)$ , the equations have to be integrated for a complete period of  $\mathbf{K}'(t)$ . After integration for one period, the state transition matrix can be used to get the correction to  $\mathbf{K}'(0)$ . Using these corrected values, the equations have to be integrated once more for a complete period to get the power spectral density. In the MFT technique, the signal  $\mathbf{K}'(t)$  is sampled at the clock frequency to obtain a discrete signal that is independent of the clock fundamentals. The number of samples required depends upon the number of Fourier coefficients needed to represent the envelop. This method is very efficient in this

case, since the envelop is a single tone. If  $\mathbf{K}'(t)$  is sampled at  $t = 0$  and  $t = T_c (= 1/f_c)$ , we get:

$$\mathbf{K}'(T_c) = e^{j\omega_m T_c} \mathbf{K}'(0) \quad (10)$$

If  $\Phi(\mathbf{K}'(t_i), t_i, t_f)$  is the state transition function for the set of equations (6), we have:

$$\mathbf{K}'(T_c) = \Phi(\mathbf{K}'(0), 0, T_c) \quad (11)$$

Equations (10) and (11) can be combined to give:

$$\mathbf{F} = \mathbf{K}'(0)e^{j\omega_m T_c} - \Phi(\mathbf{K}'(0), 0, T_c) = 0 \quad (12)$$

Since the differential equations for the cross-spectral densities are linear, the state transition function is a linear function of the initial state  $\mathbf{K}'(0)$ . Therefore, if the Newton method is used to solve the set of equations (12), the corrected initial state can be computed as:

$$\mathbf{K}'(0) = (e^{j\omega_m T_c} \mathbf{I}_N - \mathbf{T}_\phi)^{-1} \mathbf{K}'(T_c) \quad (13)$$

where  $\mathbf{I}_N$  is an  $N \times N$  identity matrix and  $\mathbf{T}_\phi$ , the sensitivity matrix, is given by [15]:

$$\mathbf{T}_\phi = \prod_{i=1}^M (\mathbf{I}_N - 0.5h_i \mathbf{A}(t_i))^{-1} (\mathbf{I}_N + 0.5h_i \mathbf{A}(t_i - h_i)) \quad (14)$$

It is assumed that the initial guess is zero and trapezoidal rule is used for the integration.

Once the corrected value of  $\mathbf{K}'(0)$  is obtained, the power spectral density can easily be computed. From equations (4), (5) and (9), it can be seen that the only term that contributes to the power spectral density is the Fourier coefficient  $d_0$  of the cross-spectral density of the output node  $K'_N(t) = E\{x_{nN}(t)X'(t, \omega)^*\}$ . This is computed as:

$$d_0 = \frac{1}{T_c} \int_0^{T_c} K'_N(t) e^{-j\omega_m t} dt \quad (15)$$

In order to do this integration, the set of differential equations (6) have to be integrated once more for one clock cycle using the corrected initial state. Once  $d_0$  is obtained, it is seen from equations (4) and (5) that the PSD is given by

$$PSD = d_0 + d_0^* = 2Re(d_0) \quad (16)$$

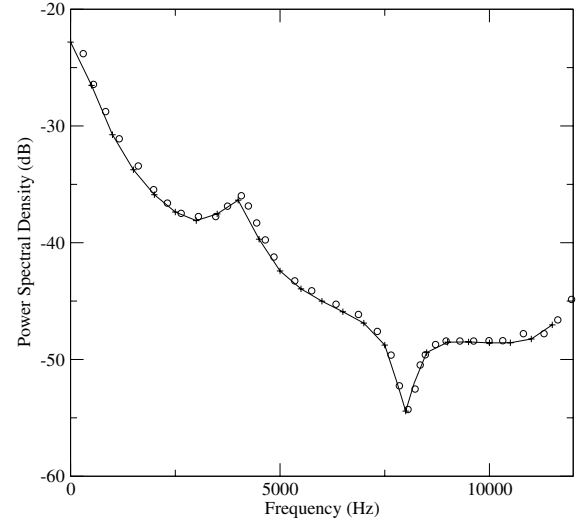
Inspection of equation (9) reveals why this technique is very efficient for spectral density computations. Other than the clock fundamentals,  $\mathbf{K}'(t)$  contains only the fundamental at  $\omega_m$ . As a result, the number of equations to be solved remains the same as the original set of differential equations. Moreover, as seen from equations (11) to (13), we need to integrate only over one clock cycle to get the corrected value of  $\mathbf{K}'(0)$ . Therefore, irrespective of  $\omega_m$ , the MFT method requires only two integrations over a clock period to compute the PSD.

## 4. RESULTS

The noise spectrum of a switched capacitor low pass and a bandpass filter was computed using the method described. For the low pass filter, computed and experimental results are available in [11]. The experimental results are obtained with an external white noise source with a PSD of -61.5dB connected to the non-inverting input of the opamp. For the bandpass filter, the noise spectral density was computed by Toth *et al* [12] and Yuan *et al* [5]. In order to compare with published results, we have used the same macromodels and

element values that were used to obtain the published results. Besides direct integration, both the shooting Newton and MFT techniques were implemented for these circuits. The code was written in Python, a public domain scripting language. It is run on a 1.7GHz Intel PIV processor running Linux. Within Python, the Lapack libraries were used.

Figure 1 shows a comparison with experimental results for the low pass filter. Both the shooting Newton and MFT give virtually identical results for the power spectral densities. The time taken by the two methods is shown in Table 1. It is seen that MFT takes less than a fourth of the time taken by the shooting Newton technique. The output noise



**Figure 1: Comparison between experimental and simulated results. o : experimental data[11], Solid line: MFT method, + : Shooting Newton.**

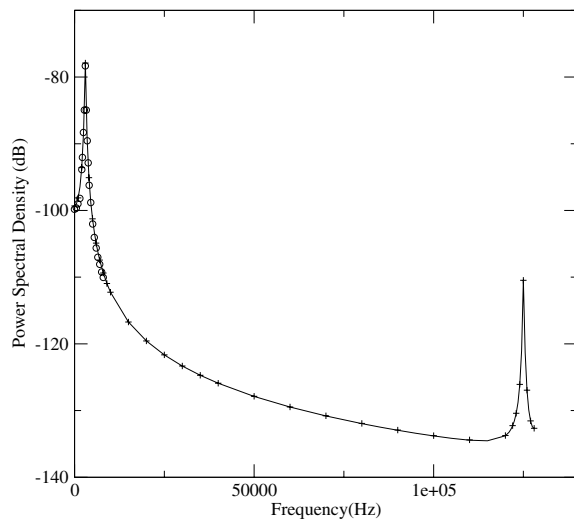
spectrum of the bandpass filter, computed using both the shooting Newton and MFT technique, is shown in figure 2. Almost identical results for the PSD are obtained using

Circuit	Method	Number of frequencies	Total time (seconds)	Time/freq (seconds)
LPF	MFT	26	464.42	17.86
	S-N	28	2250.87	80.38
BPF	MFT	53	1011	19.07
	S-N	54	45731	846.87

**Table 1: Time taken by the two methods for the lowpass(LP) and bandpass(BP) filter**

both techniques. As expected, the MFT technique is far more efficient. Results are summarised in Table 1. The time taken by MFT per frequency remains almost the same as for the low pass filter, whereas the shooting Newton method takes an order of magnitude longer. Direct integration of the equations takes more than a day.

In the table, the time taken per frequency is an average time. For the MFT method, this also represents approximately the time taken at each frequency. In the case of the shooting Newton technique, the time taken for computation depends strongly on the measurement frequency. For example, in the two cases simulated, it takes twice the amount of time to get the PSD at 0.5 kHz than at 1 kHz.



**Figure 2: Output noise spectral density of the band-pass filter.  $\circ$  : published data[12]. Solid line:MFT method,  $+$  : Shooting Newton**

## 5. CONCLUSIONS

In this paper, the possibility of using the MFT technique to speed up computations of the noise spectral density was explored. It turns out that it is extremely efficient and requires only two integrations over a clock period for each frequency at which the spectrum is desired. Unlike the shooting Newton technique, the time required is also relatively independent of the measurement frequency.

The main advantage of this method to compute the noise spectral density is that it can be integrated relatively easily into a standard circuit simulator such as SPICE. Although only switched capacitor circuits were considered here, the algorithm for computation of noise spectral density can be used for all circuits in which noise can be regarded as a perturbation, even those in which the noise is non-stationary [17]. Both the variance and the PSD of noise can be obtained. Because the spectrum is obtained based on time-varying cross-correlations, the relative contribution of various portions of the circuit to the output noise spectrum can be obtained. However, computationally it is restricted to relatively small blocks, mainly because for an  $N$  node circuit,  $\frac{N(N+1)}{2}$  equations have to be solved to get the time-varying covariance matrix. Also  $1/f$  can only be included if appropriate filtering networks are used [1, 18]. However, it is not clear how efficient this is, since it adds to the equations already present.

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