Waveform Relaxation of Linear Integral-Differential Equations for Circuit Simulation *

Yao-Lin Jiang[1,2] Omar Wing[2]

[1]*Institute of Information and System Sciences, School of Science, Xi'an Jiaotong University Xi'an, P. R. China. E-mail: yljiang@xjtu.edu.cn*

[2]*Department of Information Engineering, the Chinese University of Hong Kong Hong Kong, P. R. China. E-mail: owing@ie.cuhk.edu.hk*

Abstract

We present waveform relaxation of linear integral-differential equations which occur in circuit simulation. We give sufficient conditions for convergence and numerical experiments to verify the theoretical results.

1 Introduction

In circuit simulation, if a strict nodal formulation is used, the circuit equations after linearization are integral-differential equations of the form:

$$D\int_0^t x(\tau)d\tau + \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{dx}{dt}(t) + \begin{bmatrix} A & B \\ C & N \end{bmatrix} x(t) = f(t), \quad x_1(0) = x_{10}, \quad t \in [0,T] \tag{1}$$

where $D \in \mathbf{R}^{n\times n}$, $M, A \in \mathbf{R}^{n_1\times n_1}$, $B \in \mathbf{R}^{n_1\times n_2}$, $C \in \mathbf{R}^{n_2\times n_1}$, $N \in \mathbf{R}^{n_2\times n_2}$ such that M and N are nonsingular matrices, $x(t) = [x_1(t), x_2(t)]^t \in \mathbf{R}^n$ and $f(t) = [f_1(t), f_2(t)]^t \in \mathbf{R}^n$ in which $x_1(t), f_1(t) \in \mathbf{R}^{n_1}$ and $x_2(t), f_2(t) \in \mathbf{R}^{n_2}$ where $n_1 + n_2 = n$.

The waveform relaxation (WR) method was first presented in 1982 [1]. Recent results on accelerated techniques and convergence conditions are reported in [2 - 5].

2 Waveform Relaxation Algorithm

In this paper, we let $\tilde{M}_q = \begin{bmatrix} M_q & 0 \\ 0 & 0 \end{bmatrix}$ and $\tilde{A}_q = \begin{bmatrix} A_q & B_q \\ C_q & N_q \end{bmatrix}$ $(q = 1,2)$ where $M = M_1 - M_2$, $A = A_1 - A_2$, $B = B_1 - B_2$, $C = C_1 - C_2$, and $N = N_1 - N_2$ in which M_1 and N_1 are nonsingular matrices. The general form of the waveform relaxation algorithm with initial iteration $x^{(0)}(\cdot)$ for System (1) $(k = 1,2,\cdots)$ is

$$\begin{aligned} D_1\int_0^t x^{(k)}(\tau)d\tau + \tilde{M}_1\frac{dx^{(k)}}{dt}(t) + \tilde{A}_1 x^{(k)}(t) = \\ D_2\int_0^t x^{(k-1)}(\tau)d\tau + \tilde{M}_2\frac{dx^{(k-1)}}{dt}(t) \\ +\tilde{A}_2 x^{(k-1)}(t) + f(t), \quad x_1^{(k)}(0) = x_{10}, \quad t \in [0,T] \end{aligned} \tag{2}$$

where $D = D_1 - D_2$.

Theorem 1 The waveform relaxation solution of System (1) according to the splitting of (2) will converge if

$$\rho(M_1^{-1}M_2) < 1 \quad \text{and} \quad \rho(N_1^{-1}N_2) < 1 \tag{3}$$

Proof Let $y^{(k)}(t) = D_1\int_0^t x^{(k)}(\tau)d\tau - D_2\int_0^t x^{(k-1)}(\tau)d\tau$, thus on $[0,T]$ the algorithm (2) can be writen as $(k = 1,2,\cdots)$

$$\left\{ \begin{aligned} & y^{(k)}(t) + \tilde{M}_1\frac{dx^{(k)}}{dt}(t) + \tilde{A}_1 x^{(k)}(t) = \tilde{M}_2\frac{dx^{(k-1)}}{dt}(t) \\ & \qquad +\tilde{A}_2 x^{(k-1)}(t) + f(t), \quad x_1^{(k)}(0) = x_{10}, \\ & \frac{dy^{(k)}}{dt}(t) - D_1 x^{(k)}(t) = -D_2 x^{(k-1)}(t), \quad y^{(k)}(0) = 0 \end{aligned} \right.$$

If we denote that $y^{(k)}(t) = [y_1^{(k)}(t), y_2^{(k)}(t)]^t$ where $y_1^{(k)}(t) \in \mathbf{R}^{n_1}$ and $y_2^{(k)}(t) \in \mathbf{R}^{n_2} (k = 0,1,\cdots)$ and $D_q = [L_q, R_q]$ where $L_q \in \mathbf{R}^{n\times n_1}$ and $R_q \in \mathbf{R}^{n\times n_2} (q = 1,2)$, and $E_1 = [I_{n_1\times n_1}, 0] \in \mathbf{R}^{n_1\times n}$ and $E_2 = [0, I_{n_2\times n_2}] \in \mathbf{R}^{n_2\times n}$ where $I_{m\times m} \in \mathbf{R}^{m\times m}$ represents the identity matrix. On $[0,T]$, for any fixed k we can express the above formula with

*This work was supported by the Chinese University of Hong Kong under a special postdoctoral fellowship scheme and by the Hong Kong Research Grants Council grant CUHK 4147/97E and by the National Natural Science Fundation of China NSFC 19801027.

$[x_1^{(k)}(0), y^{(k)}(0)]^t = [x_{10}, 0]^t$ as

$$\left\{\begin{array}{l} \begin{bmatrix} M_1 & 0 \\ 0 & I_{n\times n} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x_1^{(k)}(t) \\ y^{(k)}(t) \end{bmatrix} + \begin{bmatrix} A_1 & E_1 \\ -L_1 & 0 \end{bmatrix} \\ \times \begin{bmatrix} x_1^{(k)}(t) \\ y^{(k)}(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ -R_1 \end{bmatrix} x_2^{(k)}(t) = \begin{bmatrix} M_2 & 0 \\ 0 & 0 \end{bmatrix} \\ \times \frac{d}{dt} \begin{bmatrix} x_1^{(k-1)}(t) \\ y^{(k-1)}(t) \end{bmatrix} + \begin{bmatrix} A_2 & 0 \\ -L_2 & 0 \end{bmatrix} \begin{bmatrix} x_1^{(k-1)}(t) \\ y^{(k-1)}(t) \end{bmatrix} \\ \quad + \begin{bmatrix} B_2 \\ -R_2 \end{bmatrix} x_2^{(k-1)}(t) + \begin{bmatrix} f_1(t) \\ 0 \end{bmatrix}, \\ [C_1, E_2] \begin{bmatrix} x_1^{(k)}(t) \\ y^{(k)}(t) \end{bmatrix} + N_1 x_2^{(k)}(t) = [C_2, 0] \\ \times \begin{bmatrix} x_1^{(k-1)}(t) \\ y^{(k-1)}(t) \end{bmatrix} + N_2 x_2^{(k-1)}(t) + f_2(t) \end{array}\right. \tag{4}$$

Based on the form (4), by use of the same approach in [4 - 5] we can write the algorithm (2) as an iterative process of operator equations with initial iteration $w^{(0)}(\cdot)$ in $C([0,T];\mathbf{R}^{2n})$ as follows,

$$w^{(k)}(t) = (\mathcal{R}w^{(k-1)})(t) + \varphi(t), \quad k = 1, 2, \cdots \tag{5}$$

where $w^{(k)}(t) = [x_1^{(k)}(t), y^{(k)}(t), x_2^{(k)}(t)]^t$ for any fixed k, $\varphi(t) \in \mathbf{R}^{2n}$ on $[0,T]$ and $\mathcal{R} : C([0,T];\mathbf{R}^{2n}) \mapsto C([0,T];\mathbf{R}^{2n})$ is a bounded linear operator. Further, the spectrum of the operator $\mathcal{R}$ in $C([0,T];\mathbf{R}^{2n})$ is

$$\sigma(\mathcal{R}) = \sigma(M_1^{-1}M_2) \cup \sigma(N_1^{-1}N_2) \tag{6}$$

The above relation implies that the iterative algorithm (2) converges to the solution of System (1). This completes the proof of Theorem 1.

3 Waveform Krylov Subspace Algorithm

From (5), we can similarly write System (1) as an operator equation in $L^2([0,T];\mathbf{R}^{2n})$ as follows,

$$(\mathcal{I} - \mathcal{R})w = \varphi \tag{7}$$

where $w(t) = [x_1(t), y(t), x_2(t)]^t \in \mathbf{R}^{2n}$ in which $y(t) = D\int_0^t x(\tau)d\tau$ on $[0,T]$. The spectrum of the operator $\mathcal{R}$ in $L^2([0,T];\mathbf{R}^{2n})$ also satisfies the condition (6).

We discuss the waveform GMRES which is a waveform Krylov subspace algorithm. The operator-function product $p(t) = (\mathcal{I} - \mathcal{R})w(t)$ is computed by:

1. Solve the following system for $x^\wedge(t) = [x_1^\wedge(t), x_2^\wedge(t)]^t$ on $[0,T]$:

$$D_1\int_0^t x^\wedge(\tau)d\tau + \tilde{M}_1\frac{dx^\wedge}{dt}(t) + \tilde{A}_1 x^\wedge(t) = D_2\int_0^t x(\tau)d\tau$$
$$+\tilde{M}_2\frac{dx}{dt}(t) + \tilde{A}_2 x(t), \quad M_1 x_1^\wedge(0) = M_2 x_1(0)$$

2. Set $p(t) = w(t) - w^\wedge(t)$ where $w^\wedge(t) = [x_1^\wedge(t), y^\wedge(t), x_2^\wedge(t)]^t$ in which $y^\wedge(t) = D\int_0^t x^\wedge(\tau)d\tau$.

The initial residual of Eq. (7) can be expressed as $r^{(0)}(t) = (\mathcal{R}w^{(0)} + \varphi)(t) - w^{(0)}(t)$ on $[0,T]$. The procedure computing $r^{(0)}(t)$ in which $w^{(0)}(t) = [x_1^{(0)}(t), y^{(0)}(t),\ x_2^{(0)}(t)]^t$ with $x_1^{(0)}(0) = x_{10}$ where $y^{(0)}(t) = D\int_0^t x^{(0)}(\tau)d\tau$ is given by:

1. Solve the following system for $x^\wedge(t) = [x_1^\wedge(t), x_2^\wedge(t)]^t$ on $[0,T]$:

$$D_1\int_0^t x^\wedge(\tau)d\tau + \tilde{M}_1\frac{dx^\wedge}{dt}(t) + \tilde{A}_1 x^\wedge(t) = D_2\int_0^t x^{(0)}(\tau)d\tau$$
$$+\tilde{M}_2\frac{dx^{(0)}}{dt}(t) + \tilde{A}_2 x^{(0)}(t) + f(t), \quad x_1^\wedge(0) = x_{10}$$

2. Set $r^{(0)}(t) = w^\wedge(t) - w^{(0)}(t)$ where $w^\wedge(t) = [x_1^\wedge(t), y^\wedge(t), x_2^\wedge(t)]^t$ in which $y^\wedge(t) = D\int_0^t x^\wedge(\tau)d\tau$.

Algorithm – WGMRES

1. Start: Set $r^{(0)} = \varphi - (\mathcal{I} - \mathcal{R})w^{(0)}$, $v_1 = r^{(0)}/\|r^{(0)}\|$
2. Iterate: For $l = 1, 2, \ldots$, until satisfied do:
   $h_{j,l} = \langle(\mathcal{I} - \mathcal{R})v_l, v_j\rangle$, $j = 1, 2, \ldots, l$
   $\hat{v}_{l+1} = (\mathcal{I} - \mathcal{R})v_l - \sum_{j=1}^{l} h_{j,l}v_j$
   $h_{l+1,l} = \|\hat{v}_{l+1}\|$
   $v_{l+1} = \hat{v}_{l+1}/h_{l+1,l}$
3. Form the approximate solution:
   $w^{(k)} = w^{(0)} + V_k a_k$.

In Algorithm, $V_k = [v_1, v_2, \ldots, v_k]$ and $a_k \in \mathbf{R}^k$ minimizes $\|\beta e_1^{k+1} - H_k a\|$ over $\mathbf{R}^k$ where $a \in \mathbf{R}^k$ (namely, minimizes $\|r^{(0)} - (\mathcal{I} - \mathcal{R})w\|$ over $K_k = span\{v_1, \cdots, v_k\}$) such that $e_1^{k+1} = [1, 0, \ldots, 0]^t \in \mathbf{R}^{k+1}$, $\beta = \|r^{(0)}\|$ and H_k is a matrix with dimensions $(k+1)\times k$. If $(\mathcal{I} - \mathcal{R})$ has bounded inverse and one is in the unbounded component of the complement of $\sigma(\mathcal{R})$, then it will converge to the solution of Eq. (7) (see [4]).

4 Discrete-time Case

In this section, we only discuss the p-step constant stepsize BDF method approximating the algorithm (2) [6]. The method consists of replacing $\frac{dx^{(l)}}{dt}(t)$ $(l = k, k-1)$ by the derivative of a polynomial which interpolates the computed solution at $p+1$ times $t_n, t_{n-1}, \cdots, t_{n-p}$, i.e., $\frac{1}{h}\sum_{j=0}^{p}\alpha_j x_{n-j}^{(l)}(l = k, k-1)$ where α_j $(j = 0, 1, \cdots, p)$ are the coefficients of a BDF method. Further, we replace $\int_0^t x^{(l)}(\tau)d\tau$ by

$h\sum_{j=0}^{n-1} x_{n-j}^{(l)}$ at time point t_n $(l = k, k-1)$. Thus, the discrete-time form of the algorithm (2) is

$$\begin{cases} hD_1 \sum_{j=0}^{n-1} x_{n-j}^{(k)} + \frac{1}{h}\sum_{j=0}^{p} \alpha_j \tilde{M}_1 x_{n-j}^{(k)} + \tilde{A}_1 x_n^{(k)} = \\ hD_2 \sum_{j=0}^{n-1} x_{n-j}^{(k-1)} + \frac{1}{h}\sum_{j=0}^{p} \alpha_j \tilde{M}_2 x_{n-j}^{(k-1)} + \tilde{A}_2 x_n^{(k-1)} + f_n, \\ t \in [0,T], \quad n = p, p+1, \cdots, p' \end{cases} \tag{8}$$

where $f_n = [f_n^1, f_n^2]^t$ and for any $k \geq 1$ the values $x_n^{(k)} (= x_n^{(0)})$ are known for $n = 0, 1, \cdots, p-1$, and the values $x_n^{(k)}$ are unknown for $n = p, p+1, \cdots, p'$ where $t_{p'} = T$.

Denote $-hD\sum_{k=1}^{p-1} x_{p-k}^{(0)} = [g_1^h, g_2^h]^t$ and $hD_q = \begin{bmatrix} D_{11}^q(h) & D_{12}^q(h) \\ D_{21}^q(h) & D_{22}^q(h) \end{bmatrix}$ $(q = 1, 2)$. Now let $\Phi^{(l)} = [\phi_p^{(l)}, \cdots, \phi_{p'}^{(l)}]^t$ and $\Psi^{(l)} = [\psi_p^{(l)}, \cdots, \psi_{p'}^{(l)}]^t$ where $x_n^{(l)} = [\phi_n^{(l)}, \psi_n^{(l)}]^t$ in which $\phi_n^{(l)} \in \mathbf{R}^{n_1}$ and $\psi_n^{(l)} \in \mathbf{R}^{n_2} (l = k, k-1)$ for $n = 0, 1, \cdots, p'$. Let also $F_1 = [-\sum_{j=1}^{p} \alpha_j M \phi_{p-j}^{(0)} + h(g_1^h + f_p^1), -\sum_{j=2}^{p} \alpha_j M \phi_{p+1-j}^{(0)} + h(g_1^h + f_{p+1}^1), \cdots, -\alpha_p M \phi_{p-1}^{(0)} + h(g_1^h + f_{2p-1}^1), h(g_1^h + f_{2p}^1), \cdots, h(g_1^h + f_{p'}^1)]^t$ and $F_2 = [(g_2^h + f_p^2), \cdots, (g_2^h + f_{p'}^2)]^t$. Denote also the Kronecker product of two matrices A and B by $A \otimes B$. Now, for any fixed k we can compactly write (8) as

$$\begin{bmatrix} \Phi^{(k)} \\ \Psi^{(k)} \end{bmatrix} = \begin{bmatrix} X_{11}^1 & X_{12}^1 \\ X_{21}^1 & X_{22}^1 \end{bmatrix}^{-1} \left(\begin{bmatrix} X_{11}^2 & X_{12}^2 \\ X_{21}^2 & X_{22}^2 \end{bmatrix} \begin{bmatrix} \Phi^{(k-1)} \\ \Psi^{(k-1)} \end{bmatrix} + \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \right) \tag{9}$$

in which $X_{11}^q = M_\alpha \otimes M_q + hI \otimes (A_q + D_{11}^q(h)) + hL \otimes D_{11}^q(h)$, $X_{12}^q = hI \otimes (B_q + D_{12}^q(h)) + hL \otimes D_{12}^q(h)$, $X_{21}^q = I \otimes (C_q + D_{21}^q(h)) + L \otimes D_{21}^q(h)$, and $X_{22}^q = I \otimes (N_q + D_{22}^q(h)) + L \otimes D_{22}^q(h)$ $(q = 1, 2)$ where $I \in \mathbf{R}^{s \times s}$ and $L \in \mathbf{R}^{s \times s}$ is a strictly low triangle matrix such that $L_{ij} = 1 (i > j)$ and $M_\alpha \in \mathbf{R}^{s \times s}$ is a low triangle matrix where $s = p' - p + 1$. The proof of the following theorem is nearly the same as one in [4]. For brevity, we shall omit it.

Theorem 2 When the condition (3) is satisfied, the discrete-time waveform relaxation iteration process (8) always converges for small enough time-step h.

5 Numerical Experiments

In this section, we present numerical experiments based on a linear circuit shown in Figure 1. The system of circuit equations has a form as System (1) in which the algebraic part does not exist (namely, the matrices B, C, N and the function $f_2(t)$ are nil). In this example, the matrices D, M, and A respectively are

$$D = \begin{bmatrix} \frac{1}{L_1} + \frac{1}{L_2} & -\frac{1}{L_2} & 0 & 0 & 0 \\ -\frac{1}{L_2} & \frac{1}{L_2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{L_3} + \frac{1}{L_4} & -\frac{1}{L_4} & 0 \\ 0 & 0 & -\frac{1}{L_4} & \frac{1}{L_4} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{L_5} \end{bmatrix}$$

and

$$M = \begin{bmatrix} c_1 & 0 & 0 & 0 & 0 \\ 0 & c_2 & -c_2 & 0 & 0 \\ 0 & -c_2 & c_2 + c_3 & 0 & 0 \\ 0 & 0 & 0 & c_4 & -c_4 \\ 0 & 0 & 0 & -c_4 & c_4 + c_5 \end{bmatrix}$$

and

$$A = \begin{bmatrix} G_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & G_2 \end{bmatrix}$$

Further, $x(t) = [v_1(t), \cdots, v_5(t)]^t$ with $x(0) = [0, \cdots, 0]^t$, $f(t) = [j_0(t), 0, \cdots, 0]^t$, $T = 100\pi$ and the input function $j_0(t) = (1 + 0.2sin(10t))sin(t) + (1 + 0.2sin(0.1t))sin(t)(0 \leq t < 20\pi)$ and $j_0(t) = 0(20\pi \leq t \leq 100\pi)$.

This circuit is a band-pass filter with a center frequency of 1 rad/sec and a bandwidth of 0.05 rad/sec. The input is a pulsed amplitude-modulated signal (Figure 2). The carrier frequency is 1 rad/sec and the modulating signals are two sinusoids of freqnecies 0.1 rad/sec and 10 rad/sec. At the output, we see the effect of narrow band. The output is a series of pulsed sinusoids with decreasing amplitudes (Figure 3).

In our experiments, we let $c_1 = 12.36F$, $c_2 = 0.030902F$, $c_3 = 40F$, $c_4 = 0.030902F$, $c_5 = 12.36F$, $L_1 = 0.080906H$, $L_2 = 32.36H$, $L_3 = 0.025H$, $L_4 = 32.36H$, $L_5 = 0.080906H$, and $G_1 = G_2 = 1mho$. The basic ordinary differential equation code was the Backward Euler method. The time-step was 0.1π. The error with tolerance 1×10^{-5} was defined as the sum of the squared differences of successive waveforms taken over all time points.

The known Jacobi waveform relaxation algorithm [1] of the circuit has a form of (2) in which D_1, M_1, and A_1 respectively are the diagonal matrices of the matrices D, M, and A. In the Jacobi splitting, $\rho(M_1^{-1}M_2)$ is less than one and the process converges. Now, if we let $M_1 = 10I_{5\times5}$ and keep D_1 and A_1 as in the Jacobi splitting then $\rho(M_1^{-1}M_2)$ is large than one and the process does not converge. The experiment results on these two splittings were presented in Figure 4.

6 Conclusion

We have presented new theoretical results on the convergence of the waveform relaxation algorithm and the waveform Krylov subspace algorithm (WGMRES) for systems of linear integral-differential equations for circuit simulation. The numerical experiments here show that the splitting of matrices is crucial to convergence.

References

[1] E. Lelarasmee, A. Ruehli, and A. L. Sangiovanni-Vincentelli, "The waveform relaxation method for time-domain analysis of large scale integrated circuits," *IEEE Trans. on CAD of IC and Systems*, vol. 1, no. 3, pp. 131–145, July 1982.

[2] A. Lumsdaine, M. W. Reichelt, J. M. Squyres, and J. K. White, "Accelerated waveform methods for parallel transient simulation of semiconductor devices," *IEEE Trans. on CAD of IC and Sys.*, vol. 15, no. 7, pp. 716–726, July 1996.

[3] Y. L. Jiang and O. Wing, "Splitting techniques to speed up the convergence of waveform relaxation methods for tightly coupled circuit systems," Proceedings of 1997 European Conference on Circuit Theory and Design, pp. 1054–1058, Budapest, September 1997.

[4] Y. L. Jiang, W. S. Luk, and O. Wing, "Convergence-theoretics of classical and Krylov waveform relaxation methods for differential-algebraic equations," *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences*, vol. E80-A, no. 10, pp. 1961–1972, October 1997.

[5] Y. L. Jiang and O. Wing, "On the spectra of waveform relaxation operators for circuit equations," *IEICE Trans. Fundamentals of Electronics, Communications and Computer Sciences*, vol. E81-A, no. 4, pp. 685–689, April 1998.

[6] J. Ogrodzki, *Circuit Simulation Methods and Algorithms*, CRC Press, London, 1994.

[7] P. Linz, *Analytical and Numerical Methods for Volterra Equations*, SIAM, Philadelphia, 1985.

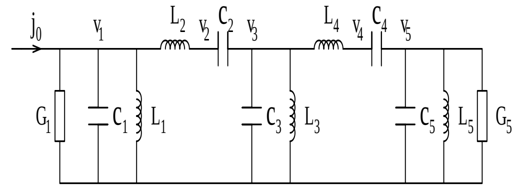

Figure 1: A linear circuit described by integral-differential equations.

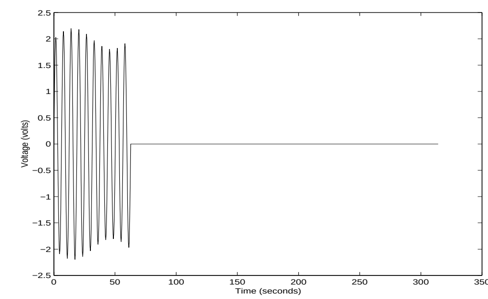

Figure 2: Waveform of the input function $j_0(t)$ in Figure 1 on $[0, 100\pi]$.

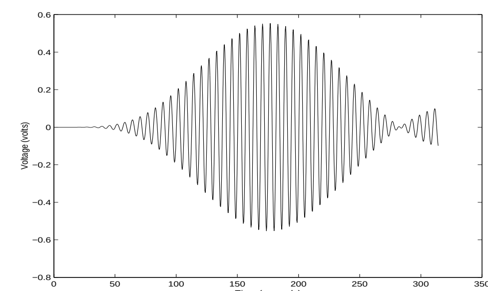

Figure 3: Waveform of the voltage $v_5(t)$ in Figure 1 on $[0, 100\pi]$.

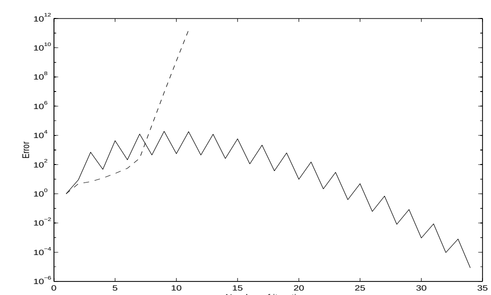

Figure 4: Waveform relaxation for the circuit in Figure 1. The case of the Jacobi splitting was shown by the solid line and the second case was shown by the dashed line.