Generating Sparse Partial Inductance Matrices with Guaranteed Stability‡

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Abstract

This paper proposes a definition of magnetic vector potential that can be used to evaluate sparse partial inductance matrices. Unlike the commonly applied procedure of discarding the smallest matrix terms, the proposed approach maintains accuracy at middle and high frequencies and is guaranteed to be positive definite for any degree of sparsity (thereby producing stable circuit solutions). While the proposed technique is strictly based upon potential theory (i.e. the invariance of potential differences on the zero potential reference choice), the technique is, nevertheless, presented and discussed in both circuit and magnetic terms. The conventional and the proposed sparse formulation techniques are contrasted in terms of eigenvalues and circuit simulation results on practical examples.

1: Introduction

When induced current return paths are unknown, circuit inductances are modeled using partial self and mutual inductances. Although partial inductances stencil into circuit graphs like conventional self and mutual inductances, the lack of sparsity in partial inductance matrices places an enormous burden, in terms of run time and storage, on the subsequent circuit simulation or equivalent circuit construction. Because partial inductances are frequently misunderstood and misused, and our proposed technique computes partial inductances from a different perspective, this paper will begin with a brief review of partial inductances and partial inductance matrices.

1.1: Partial Inductances

Partial inductances were defined long before the advent of integrated circuits. Rosa published several reports for the Bureau of Standards in the early 1900's (one relevant report on linear conductors can be found in [11]), and Grover provided a comprehensive summary of partial inductances in 1946 [1]. Partial inductances were used primarily in power engineering until 1972 when Ruehli introduced them into the world of integrated circuits [12].

First and foremost, partial inductances have no physical meaning because inductance is really a property of closed loops. Heaviside, for example, advocated the exclusive consideration of closed circuits, and as a means of securing external continuity, he proposed artificial return paths that would radially diverge from the positive terminal and radially converge on the negative terminal [2].

Partial inductances, however, provide a useful means of ascribing portions of a total loop inductance to segments along the loop. Because partial inductances behave like closed loop inductances, partial inductances are particularly well-suited for circuit simulators such as SPICE or ASTAP. They depend only on the geometry between conductor segments, which makes partial mutual inductances (like closed loop mutual inductances) reciprocal, and the constitutive branch equations for partial inductances are identical to that of closed loop inductances.

The utility of partial inductances is particularly evident when either multiconductor geometries or high frequency effects are investigated. When a multiconductor problem is described by a set of partial inductances, the sum of the partial self and partial mutual inductances along any closed loop path will yield the total loop inductance of the path. The partial mutual inductances are, of course, appropriately weighted by either ± 1 to account for the relative orientation of segment currents [10]. The same holds true for systems of more than one loop. The mutual inductance between two loops can be obtained via the appropriate sum of partial mutual inductances between loops. Furthermore, when individual conductor segments are broken up into multiple parallel conductor segments, high frequency phenomenon such as skin effect and proximity effects can be analyzed [13].

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‡ This work was supported in part by IBM and the Semiconductor Research Corporation under contract 95-DJ-343.

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Partial inductances are best understood in terms of the normalized magnetic vector potential drop along a conductor segment due to current in that, or another segment. Consider the two conductor segments, *i* and *j*, shown in Fig.1, with a current I_j in segment *j*. Because partial inductances can be defined between an arbitrary pair of conductor segments, the two segments in Fig.1 are purposely depicted as offset, of unequal length, and inclined at an angle with respect to each other. The partial self inductance L_{jj} along the segment *j* is given by

$$L_{jj} = \frac{1}{I_j\ a_j}\left[\int_{a_j}\int_{l_j} \mathbf{A}_{jj} \cdot d\mathbf{l}_j\ da_j\right] \tag{1}$$

where $\mathbf{A}_{jj}$ is the magnetic vector potential along segment *j* due to the current I_j in segment *j*, which has a cross section a_j. The partial mutual inductance M_{ij}, which relates the induced voltage drop along segment *i* due to a change in the current along segment *j*, is given by a similar expression

$$M_{ij} = \frac{1}{I_j\ a_i}\left[\int_{a_i}\int_{l_i} \mathbf{A}_{ij} \cdot d\mathbf{l}_i\ da_i\right] \tag{2}$$

In this expression, $\mathbf{A}_{ij}$ is the magnetic vector potential along segment *i* due to the current I_j in segment *j*. Segment *i* has a cross section a_i.

The magnetic vector potential $\mathbf{A}_{ij}$ is defined, as it normally is in magnetostatics, by the Coulomb gauge (i.e. $\nabla\mathbf{A} = 0$), and hence,

$$\mathbf{A}_{ij} = \frac{\mu_0}{4\pi a_j}\left[\int_{a_j}\int_{l_j} \frac{\mathbf{I}_j}{r_{ij}}\ d\mathbf{l}_j\ da_j\right] \tag{3}$$

In this expression, r_{ij} is the geometric distance between the point r_i in segment *i* and r_j in segment *j*. (1), (2), and (3) can be combined into a form which is similar to Neumann's equation for closed loops:

$$L_{ij} = \frac{\mu_0}{4\pi a_i a_j}\int_{a_i}\int_{a_j}\int_{l_i}\int_{l_j} \frac{dl_i \cdot dl_j}{r_{ij}}\ da_i da_j \tag{4}$$

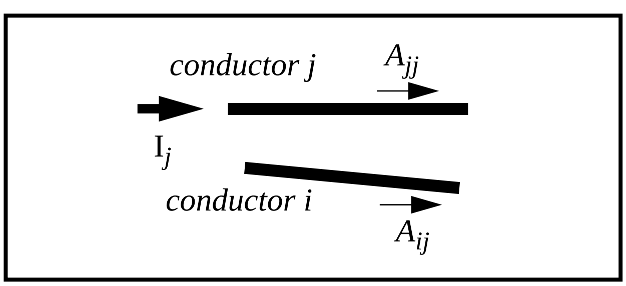

FIGURE 1: Magnetic vector potential formulation of partial-self and partial-mutual inductance.

In the Coulomb gauge, when displacement currents and dc losses are negligible, the electric field is given by the time derivative of magnetic vector potential, $E = -(dA/dt)$, and consequently, the branch voltages are given by the familiar relationship $V = L(dI/dt)$.

1.2: Partial Inductance Matrices

The partial inductance matrix for a set of *n* conductors is a *nxn* real symmetric matrix. The diagonal terms in the matrix are partial self inductances, while the off-diagonal terms are partial mutual inductances. When the partial inductance matrix is post multiplied by a vector of *n* segment currents, the resultant product yields a vector of the magnetic vector potential drops along each segment:

$$\begin{bmatrix} L_{11} & M_{12} & \dots \\ M_{12} & L_{22} & \dots \\ : & : & L_{nn} \end{bmatrix} \begin{bmatrix} i_1 \\ : \\ i_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n}\left(\int A_{1i} \cdot dl_1\right) \\ : \\ \sum_{i=1}^{n}\left(\int A_{ni} \cdot dl_n\right) \end{bmatrix} \tag{5}$$

As mentioned previously, it is well known that partial inductance matrices are dense and positive semi-definite[7]. Moreover, it is understood that making the matrix sparse by merely discarding the smallest terms can render the matrix indefinite and thereby introduce positive pole(s) in subsequent circuit simulations (more on this later).

One less known characteristic, however, is that for special problems a constant can be subtracted from every non-zero element in the matrix without altering the circuit solution[9]. In these cases, a more accurate diagonally dominant, and hence stable, sparse matrix can be formulated (relative to the common procedure of discarding or truncating the smallest matrix terms) by first shifting all the non-zero matrix terms and then truncating the smallest matrix terms [4].

This technique can be demonstrated with a simple example. Consider the system in Fig.2, where two isolated subcircuits are connected by two long conductors such as

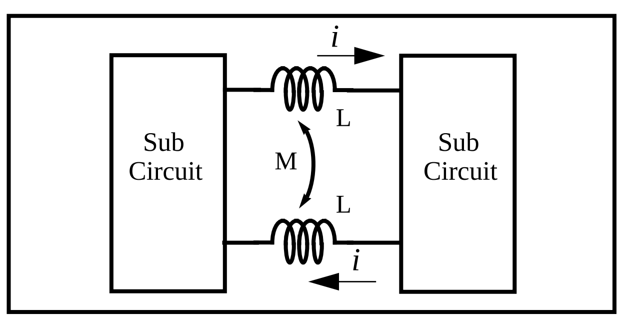

FIGURE 2: Two isolated sub-circuits connected via two long conductors. KCL constrains inductor currents to be equal and opposite.

the two conductors on a twin lead transmission line. If these two conductors are identical, the partial inductance matrix for this example is a simple 2x2 symmetric matrix,

$$\begin{bmatrix} L & M \\ M & L \end{bmatrix} \tag{6}$$

The eigenvalues of this matrix are L-M and L+M, and the associated eigenvectors are [1,-1] and [1,1] respectively. Because the circuit simulator is constrained to satisfy KCL, only the first eigenvalue and eigenvector will be present in the circuit solution. The second eigenvalue and eigenvector represent a solution for which both wires have their return currents at infinity.

The individual terms within this matrix are relatively unimportant, since it is the difference between terms that matters. Adding or subtracting a constant from every matrix element will not affect the differential eigenvalue, L-M, and, therefore, will not affect the circuit solution. It will affect the L+M eigenvalue, but since there are no return currents at infinity, the circuit simulation is unaffected. The shifted and truncated sparse matrix $\begin{bmatrix} (\mathbf{L}-\mathbf{M}) & \mathbf{0} \\ \mathbf{0} & (\mathbf{L}-\mathbf{M}) \end{bmatrix}$ (found by subtracting M from all terms in (6)) will provide an accurate circuit solution, whereas the truncated sparse matrix $\begin{bmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} \end{bmatrix}$ will not.

2: Sparse Partial Inductance Matrix Formulation

While the preceding circuit arguments are intriguing (many partial inductance matrices can indeed be shifted by a constant without altering the circuit solution), subtracting a constant from every non-zero term in the matrix is not a general purpose first step in formulating a sparse partial inductance matrix. If the original matrix models a system of conductors of varying lengths or non-orthogonal orientations, subtracting a constant from every element in the matrix will alter the “differential” eigenvalues and eigenvectors of the original matrix and thereby change the circuit solution.

To properly account for these generalities when formulating a partial inductance matrix, our approach begins with a redefinition of the magnetic vector potential. We no longer assume that conductor segment currents return at infinity. All incremental currents are assumed to return at a finite and constant radius r_o from their origin. (For example, r_o may represent the physical package size). A cross-section view of this assumption is depicted in Fig.3. The definition in (3), which describes the magnetic vector potential along conductor segment i due to an isolated current in conductor segment j, is replaced by

$$\mathbf{A}_{ij} = \frac{\mu_o \mathbf{I}_j}{4\pi} \int_{l_j} f\,(r_{ij}, r_o)\, dl_j \tag{7}$$

where

$$\begin{aligned} f\,(r_{ij}, r_o) &= \left(\frac{1}{r_{ij}} - \frac{1}{r_o}\right), & r_o \ge r_{ij} \\ &= 0 \quad , & r_o < r_{ij}\ . \end{aligned} \tag{8}$$

r_o is the finite and constant radius depicted in Fig.3. The reader may, at this point, note the similarities between (8) and the electrostatic potentials inside and outside a spherical capacitor.

When this new definition of magnetic vector potential is applied to (1) and (2), the changes to the original partial self and mutual inductances L_{ii} and M_{ij} are generally trivial and easily interpreted. Note, in the following paragraphs, matrix terms for the original dense matrix are denoted by L_{ii} and M_{ij}, while terms for the sparse, shift-and-truncate matrix are denoted by $\mathcal{L}_{ii}$ and $\mathcal{M}_{ij}$. Furthermore, the separation r_{ij} represents the entire range of separations between any two points along segments i and j.

When r_o is strictly greater than or equal to the separation r_{ij}, the magnetic vector potential induced by an incremental current vector in segment j at any point along segment i is shifted by a constant (just like the electric potential inside a spherical capacitor). From (7) and (8), the shifted partial self and mutual inductances $\mathcal{L}_{ii}$ and $\mathcal{M}_{ij}$ are given by

$$\mathcal{L}_{ii} = L_{ii} - \frac{\mu_o}{4\pi r_o} dl_i^{\,2} \tag{9}$$

and

$$\mathcal{M}_{ij} = M_{ij} - \frac{\mu_o}{4\pi r_o} dl_i \cdot dl_j . \tag{10}$$

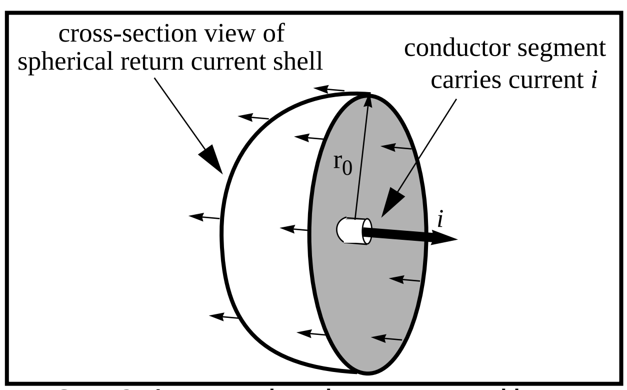

FIGURE 3: Incremental conductor segment with current *i* and a cut away view of the spherical shell of return current -*i* at a radius r_0.

When r_o is strictly less than or equal to the separation r_{ij}, the magnetic vector potential induced by an incremental current vector in segment *j* at any point along segment *i* is zero (just like the electric potential outside a spherical capacitor). Therefore, from (7) and (8) the shifted partial mutual inductances M_{ij} and M_{ji} are also zero.

When the range of r_{ij} overlaps r_o, (i.e. r_{ij} (min) < r_o < r_{ij} (max)), the definition of magnetic vector potential offered in (7) and (8) produces a non-trivial change in the partial inductance terms L_{ii} and M_{ij}. While an approximate determination of L_{ii} and M_{ij} can be obtained by numerically integrating first (7), and then (1) or (2), this is unnecessary in practice.

For example, when the radius r_o is much greater than both the segment lengths, l_i and l_j, and the segment cross sections, a_i and a_j, the shifted and truncated partial mutual inductance M_{ij} can be accurately approximated using (10) only. That is, when (10) yields a positive number, M_{ij} is simply given by that result. But when (10) yields a negative number, M_{ij} is set to zero. Moreover, when the conductor segments are parallel and r_o is still greater than the conductor cross sections a_i and a_j, exact filament equations can be used to approximate the shifted partial mutual inductances L_{ii} and M_{ij}, regardless of the segment lengths l_i and l_j [5].

Interesting observations can be made at this point. First, because $\Delta M_{ij} = \Delta M_{ji}$, the proposed sparse partial inductance matrix is symmetric and can be used in a circuit simulator just like the original matrix. Second, when the system is entirely comprised of orthogonal, equal length conductor segments and r_o is strictly greater than r_{ij}, this approximation will shift all non-zero terms in the original matrix by a constant (a change that is known not to affect the circuit solution in many matrices [9]). Finally, in the Appendix, the circuit solution is shown to be invariant when the circuit is properly constructed and r_o is strictly greater than r_{ij} regardless of conductor orientation and dimensions. That is, constant shifts in the vector potential will not affect the circuit solution because the circuit solution is concerned with potential differences with respect to both electric and magnetic vector potentials.

While these changes do affect the circuit solution when r_o is less than the package dimensions, their effect will be unnoticeable in the analysis of a good package design. In a good electrical package design tight current loops can be induced at high frequencies. If the largest dimension of these loops is strictly much less than r_o, the far-magnetic fields produced by the high frequency current loops are to a first approximation zero (i.e. the monopole terms in the multipole expansions are zero), and magnetic coupling to distant conductor segments can be ignored. Current loops larger than r_o are induced only at low frequencies when resistive losses dominate and inductive drops can be largely ignored. This is why the dense inductance matrices are generally unnecessary.

3: Stability of Sparse Approximation

The energy, U_B, stored in a static magnetic field is equal to the assembly energy required to establish that field and is given by

$$U_B = \frac{1}{2}\int \mathrm{j} \cdot \mathbf{A}\ dV \tag{11}$$

where the volume integral includes all regions in which j is non-zero [8]. For a finite system of stationary currents, this assembly energy can be related to the volume integral taken over all space, of the magnetic flux density B squared, and is therefore, guaranteed positive.

$$U_B = \frac{1}{2\mu_0}\int_\infty B^2\ dV \geq 0 \tag{12}$$

Moreover, U_B can also be related to the quadratic form of the partial inductance matrix.

For example, consider a system of N conductor segments having uniform current flow across all segments and total segment currents $i_1, i_2, \ldots i_N$. The volume integral of $\mathrm{j} \cdot \mathbf{A}$ for this system can be expressed as

$$\int_V \mathrm{j} \cdot \mathbf{A}\ dV = \sum_{i=1}^{N} i_i \int_{l_i} \hat{\mathbf{A}}_i \cdot dl_i \tag{13}$$

where $\hat{\mathbf{A}}_i$ is the average magnetic vector potential over the cross-section of segment *i*. Because the average magnetic vector potential $\hat{\mathbf{A}}_i$ along any segment *i* can be expressed as the sum of the magnetic vector potentials induced by each of the N conductor segments, (13) can be expanded as follows:

$$\int_V \mathrm{j} \cdot \mathbf{A}\, dV = \sum_{i=1}^{N} i_i \int_{l_i} \left[\sum_{j=1}^{N} \frac{\mu_0}{4\pi} \int_{l_j} \frac{i_j}{r_{ij}}\, dl_j\right] \cdot dl_i \tag{14}$$

Finally, rearranging the sums and integrals in (14) equates the volume integral of $\mathrm{j} \cdot \mathbf{A}$ with the quadratic form of the partial inductance matrix L:

$$\int_V \mathrm{j} \cdot \mathbf{A}\ dV = \sum_{i=1}^{N}\sum_{j=1}^{N} L_{ij}\, i_i i_j = \left[i^T\right]\left[L\right]\left[i\right] \tag{15}$$

Hence, the partial inductance matrix L for a system of N conductors (even when the segments do not form closed loops) is positive semi-definite.

$$\left[i^T\right]\left[L\right]\left[i\right] \geq 0 \tag{16}$$

To demonstrate the positive definiteness of the sparse approximation $\mathcal{L}$, consider a system that is comprised of N conductor segments and the N current distributions formed when spherical shells of equal and opposite return current are continuously centered with radius r_0 around each incremental current vector in the N conductor segments. Depicted in Fig.4 is a two dimensional view of the return current distribution centered around the conductor segment j. Segment j conducts current i_j, has length l_j, and runs in the z direction. The two dimensional view of the return current distribution lies in the yz plane.

When viewed from the exterior, this return current distribution resembles a capsule (i.e. a cylinder of radius r_o and length l_j with spherical caps of radius r_o at both ends). When $r_o >> l_j$, the return current distribution for segment j is concentrated near the surface of this shell and is zero over much of the interior region. When $l_j >> r_o$, the return current distribution is no longer hollow, and if segment j is filament, has a radial dependence in the xy plane at the filament center given by

$$\begin{aligned} i_{sh}(r) &= -i_{seg}\frac{r}{r_0^2}\left[1-\left(\frac{r}{r_0}\right)^2\right]^{-1/2}, \quad r < r_0 \\ &= 0 \quad , \quad r \geq r_0 . \end{aligned} \tag{17}$$

Now consider the partial inductance matrix L whose quadratic form is equal to the volume integral of $\mathrm{j} \cdot \mathbf{A}$ for the N conductor segments and N "shells" of return current.

$$\mathrm{L} = \begin{bmatrix} \mathrm{L}_{seg} & \mathrm{M} \\ \mathrm{M}^T & \mathrm{L}_{sh} \end{bmatrix} \tag{18}$$

In the matrix L, L_{seg} is the inductance matrix for the N conductor segments, L_{sh} is an inductance matrix for the N shells of return currents, and M is an NxN matrix of mutual inductances coupling these two systems. Because the total, the segment, and the return shell current distributions all represent "physically" realizable current distributions, their respective matrices L, L_{seg} and L_{sh} are all positive definite. That is, given conductor segment currents $i_{seg\ 1}$, $i_{seg\ 2}$, .. $i_{seg\ N}$ and shell currents $i_{sh\ 1}$, $i_{sh\ 2}$, .. $i_{sh\ N}$,

$$\begin{bmatrix} i_{seg}^T & i_{sh}^T \end{bmatrix} \begin{bmatrix} \mathrm{L}_{seg} & \mathrm{M} \\ \mathrm{M}^T & \mathrm{L}_{sh} \end{bmatrix} \begin{bmatrix} i_{seg} \\ i_{sh} \end{bmatrix} \geq 0 \tag{19}$$

$$\begin{bmatrix} i_{seg}^T \end{bmatrix} \begin{bmatrix} \mathrm{L}_{seg} \end{bmatrix} \begin{bmatrix} i_{seg} \end{bmatrix} \geq 0 \tag{20}$$

$$\begin{bmatrix} i_{sh}^T \end{bmatrix} \begin{bmatrix} \mathrm{L}_{sh} \end{bmatrix} \begin{bmatrix} i_{sh} \end{bmatrix} \geq 0 \tag{21}$$

The sparse partial inductance matrix $\mathcal{L}$ we propose models the magnetic vector potential drop along the N conductor segments when the effects of the N shells of return current are also considered. That is $\mathcal{L}$, which results when (8) is integrated into first (7) and then (1) and (2), can also be expressed as

$$\mathcal{L} = \mathrm{L}_{seg} - \mathrm{M} . \tag{22}$$

When the shell currents are equal and opposite to the segment currents (i.e. i_{seg} = i and i_{sh} = -i), (19) can be reordered and expanded as follows:

$$\begin{bmatrix} i^T \end{bmatrix} \begin{bmatrix} \mathrm{L}_{seg} \end{bmatrix} \begin{bmatrix} i \end{bmatrix} - 2\begin{bmatrix} i^T \end{bmatrix} \begin{bmatrix} \mathrm{M} \end{bmatrix} \begin{bmatrix} i \end{bmatrix} + \begin{bmatrix} i^T \end{bmatrix} \begin{bmatrix} \mathrm{L}_{sh} \end{bmatrix} \begin{bmatrix} i \end{bmatrix} \geq 0 \tag{23}$$

Because the segment currents represent a similar but denser current distribution than the shells of equal and opposite return current, their assembly energy (given by (20) with i_{seg} = i), is greater than the assembly energy of the shells of return current (given by (21) with i_{sh} = -i). Therefore, (20) can be substituted into (23) without changing the inequality, and hence, the positive definiteness or stability of the sparse matrix $\mathcal{L}$ is shown.

$$\begin{aligned} 2\begin{bmatrix} i^T \end{bmatrix}\begin{bmatrix} \mathcal{L} \end{bmatrix}\begin{bmatrix} i \end{bmatrix} &= 2\begin{bmatrix} i^T \end{bmatrix}\begin{bmatrix} \mathrm{L}_{seg} - \mathrm{M} \end{bmatrix}\begin{bmatrix} i \end{bmatrix} \geq \\ \begin{bmatrix} i^T \end{bmatrix} \begin{bmatrix} \mathrm{L}_{seg} \end{bmatrix} \begin{bmatrix} i \end{bmatrix} &- 2\begin{bmatrix} i^T \end{bmatrix} \begin{bmatrix} \mathrm{M} \end{bmatrix} \begin{bmatrix} i \end{bmatrix} + \begin{bmatrix} i^T \end{bmatrix} \begin{bmatrix} \mathrm{L}_{sh} \end{bmatrix} \begin{bmatrix} i \end{bmatrix} \geq 0 \end{aligned} \tag{24}$$

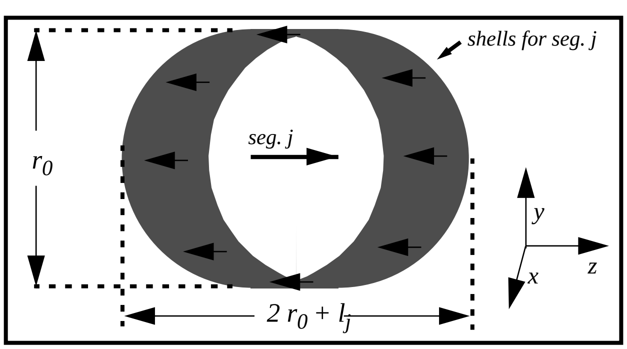

FIGURE 4: Two dimensional view of current shell *j* centered about conductor segment *j*.

4: Examples

Consider the two power planes depicted in Fig.5. When power plane inductance is modeled, power planes such as these are meshed into separate x and y conductor segments. (Even solid power planes are meshed into separate x and y conductors.) Because orthogonal conductors do not couple magnetically, the resulting inductance matrix is a block diagonal matrix. For the two planes depicted in Fig.5, we created, using FastHenry [3], a partial inductance matrix to model the magnetic coupling in the x direction. Each plane was meshed into 100 equal 10 mm square segments along the x direction, and uniform current

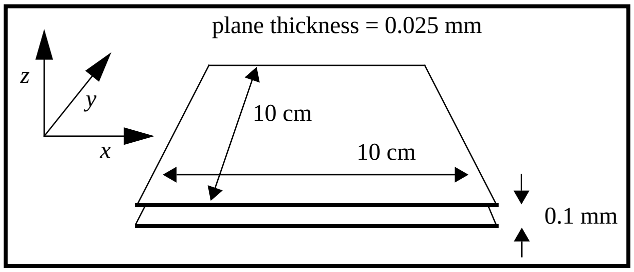

FIGURE 5: Two power planes for inductance extraction.

flow was assumed along all segments (FastHenry parameters nhinc and nwinc were set to one).

From this FastHenry partial inductance matrix, we created two sparse approximations of the full matrix. The first approximation was formed using the simple procedure proposed here. That is, we evaluated (10) with r_o=12 mm, and when the result was negative, the matrix term was set to zero. The second sparse approximation was formed by discarding all mutual inductances less than 0.75 nH. In both cases, 38,160 of the total 40,000 matrix terms were set to zero, (i.e. both approximations were >95% sparse). Finally, the eigenvalues of the full matrix and the two sparse approximations were computed. The 200 eigenvalues for each matrix are plotted in Fig.6.

In observing Fig.6, note the smallest 100 eigenvalues of the proposed approximation match those of the full matrix. Furthermore, the proposed approximation shows the same discontinuous jump between the 100th and 101st eigenvalue (the first 100 eigenvalues of this example are associated with nearly “equal and opposite” currents in the 100 pairs of vertically adjacent conductor segments). Significant differences between the proposed approximation and the full matrix are found only at the largest eigenvalues.

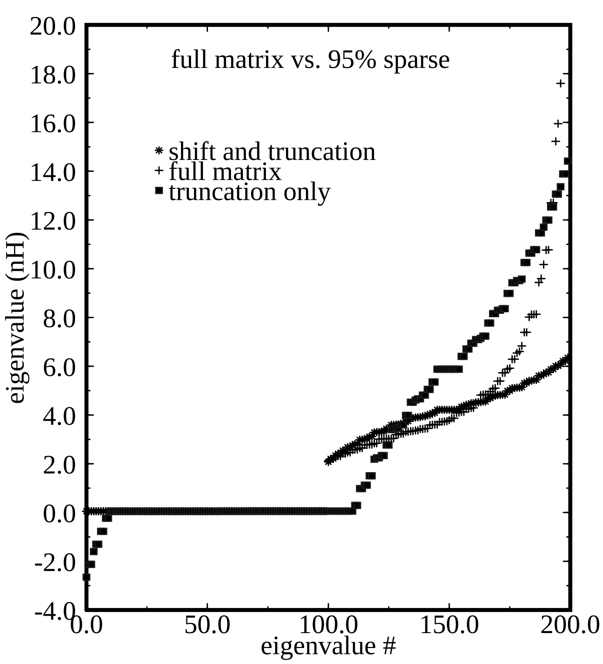

FIGURE 6: Eigenvalues for full and 95% sparse matrices modelling the two power planes in Fig.5.

These differences, however, are insignificant in most circuit simulations. The largest eigenvalues of the full matrix “model” the effective inductance associated with largest current loops. (In fact, partial inductance matrices are often irreducible non-negative matrices, and as such, the largest eigenvalue is associated with a non-negative or return-at-infinity eigenvector [6].) If smaller current loops can be formed, these larger loops only form when resistive losses dominate (i.e. $|j\omega L| << R$). Therefore, underestimating loop inductance when smaller loops are available will not affect the circuit solution.

The approach of simply discarding the smallest terms in the inductance matrix, however, yields both an inaccurate and an unstable approximation as it fails to match the eigenvalues of the full matrix at both extremes and in the middle. Although the instability will eventually vanish when either more or less sparse approximations are formed (i.e. all eigenvalues become positive), the inaccuracies at both the high and the middle frequencies persist.

For example, depicted in Fig.7 are the eigenvalues of the full matrix and the conventional and proposed sparse approximations when 31,216 of the 40,000 elements are set to zero (≈78% sparse). Again, the proposed method yields a more accurate approximation. The discontinuous jump between the 100th and 101st eigenvalue found in both the full matrix and the proposed approximation is still missing from the conventional approximation. Furthermore, the middle eigenvalues that model the effective inductance associated with moderate current loops are initially underestimated, and then finally, overestimated in the conventional approximation.

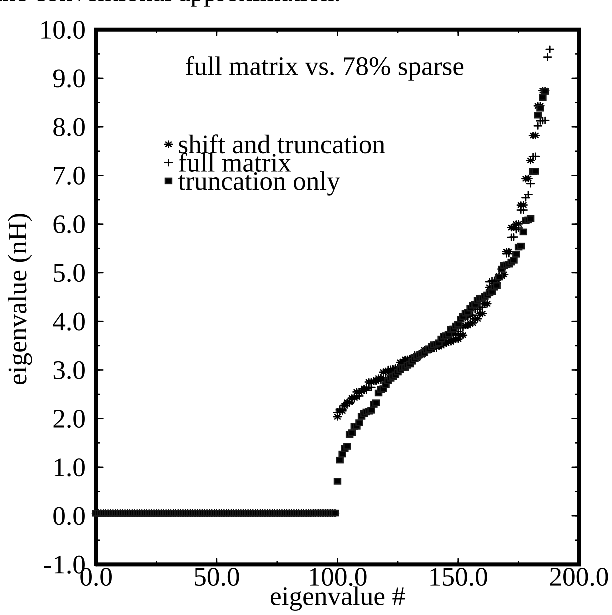

FIGURE 7: Eigenvalues for full and 78% sparse matrices modelling the two power planes in Fig.5

To further test the stability of the simplified shift-and-truncate approximation (i.e. when (9) or (10) yield a negative number, L or M in the sparse matrix is set to zero), a program was written to create random partial inductance matrices and compute the eigenvalues of the full matrix and sparse approximations thereof. We considered parallel hollow tubes because simple filament equations could be used, and more importantly, self inductances (the diagonal matrix terms) would be minimized for a worst case test of positive definiteness.

Millions of random cases involving ten conductors were created and analyzed. When the conductors did not overlap or only moderate overlap was allowed, all sparse approximations based the simple use of (9) or (10) were stable. Unstable approximations only occurred when significant conductor overlap was allowed and r_o approached the minimum separation between conductor axes.

Finally, to contrast the effects of full and sparse inductance matrices in circuit simulation, two simple circuits involving two pairs of long-parallel conductors were analyzed. These two circuits are depicted in Fig.8. In circuit A, each pair of conductors was driven separately but in opposite directions. In circuit B, one pair of conductors served as the return path for the other. Partial inductances were derived assuming all conductor cross-sections were 2x2 mm square, d_1 = 1 cm, d_2 = 4cm, and d_3 = 40 cm. Each conductor was broken into forty equal segments in order to create a large yet illustrative partial inductance matrix.

To make the inductive effects dominate, R_s and R_t were set to 1 and 10 ohms, and a slow rise time (10 ns) was considered so that capacitive coupling could be ignored for this example. Both circuits were analyzed using the full matrix and the sparse matrices. The conventional sparse matrix was formed by discarding all mutual inductances less than 245 pH. The proposed sparse matrix was formed by assuming a finite return distance (r_o in (9) and (10)) of 40.5 mm. Both matrices were equally (≈90%) sparse. Depicted in Fig.9 are the results of these simulations.

In both examples, the shift-and-truncate results more closely match the full matrix results. In circuit A, the differences were small because the current loops were small and both sparse matrices explicitly retained the nearest neighbor coupling terms. In circuit B, the differences were larger because the current loops were larger, and although neither sparse matrix explicitly retained all the dominate coupling terms, the shift-and-truncate matrix implicitly retained more coupling effects. That is, the remaining non-zero terms in the shift-and-truncate approximation assumed a finite return distance.

It should be noted however, that while the shift-and-truncate results were clearly better than the truncate-only results in circuit B, these results were worse than either approximation in circuit A. This was because tight return current paths (relative to the assumed return distance r_o) were not available in the second circuit, and consequently, the dominant coupling effects were underestimated.

5: Conclusions

Partial inductances are extremely useful in modeling circuit inductances when the induced current loops are unknown. Unfortunately, these matrices are dense and

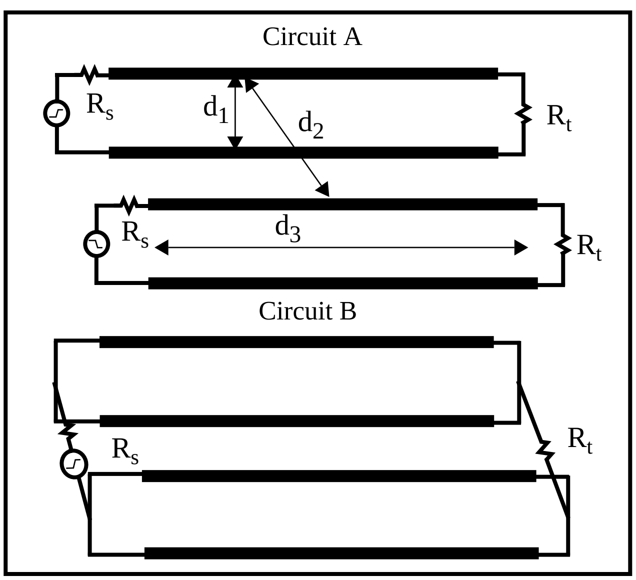

FIGURE 8: Two circuits involving parallel transmission lines.

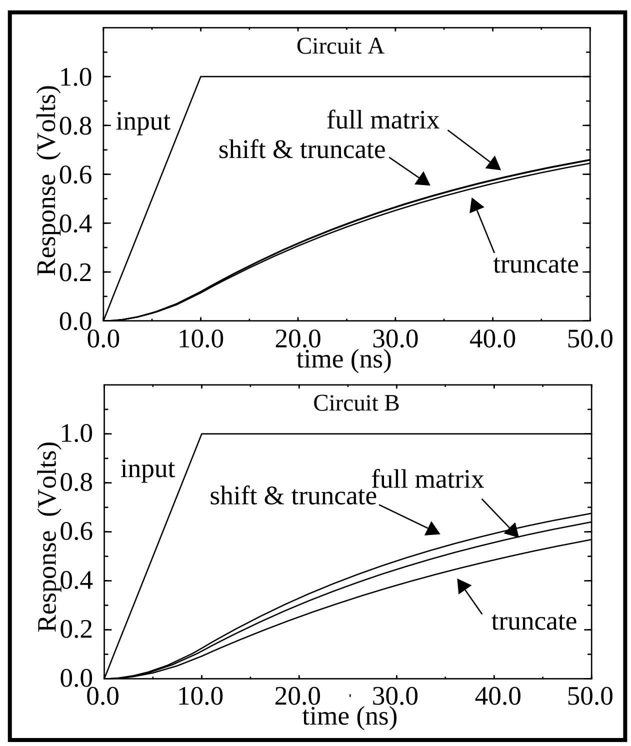

FIGURE 9: Simulation results for two circuits in Fig.8.

defy conventional simplifications (i.e. the smallest matrix cannot be indiscriminately discarded). This paper proposes a sparse formulation technique that continuously yields positive definite sparse approximations of the original inductance matrix.

Presented and discussed in both circuit and magnetic terms, these sparse approximations accurately model the high frequency behavior of the circuit because sparsity is achieved by underestimating loop inductance when the current loops are large. Provided the circuit has been well designed and properly used, such that tight current loops are possible, these inaccuracies will not affect the overall accuracy of the circuit solution. The combined circuit equations, (L, R, and C), will still accurately model all regions of operation because the larger current loops will only occur at low frequencies when resistive losses dominate. The conventional and the proposed sparse formulation techniques are contrasted with respect to matrix eigenvalues and circuit simulation results on practical examples.

Appendix

Invariance of Circuit Solution when $r_o > r_{ij}$

Although segment to segment interactions change, it can be shown that all closed-loop to closed-loop and all closed-loop to open-loop circuit interactions are invariant to the shift in vector potentials when the equivalent circuit is properly constructed and r_o is strictly greater than the maximum dimensions of the circuit. Because of its relative importance, we will show that the voltage induced across a conductor segment is unaffected by our proposed formulation of partial inductance. The invariance of closed loop self and mutual inductances and the invariance of the flux from a single conductor segment penetrating an adjacent closed loop follows from this proof.

The following equation represents the voltage induced across a conductor segment *i*, due to time varying currents in N conductor segments:

$$V_i = \sum_{j=1}^{N} M_{ij} \frac{dI_j}{dt} \quad \text{where } M_{ii} = L_{ii}\,. \tag{25}$$

If the equivalent circuit has been properly assembled (i.e. all cutsets in the dc equivalent circuit are expressible in terms of inductor currents) and displacement currents are negligible, the N segment currents will form one or more current loops. Assume for the following discussion that the N segment currents form M current loops.

Under these assumptions, when the changes from (9) and (10) are substituted into (25), the change in the branch voltage V_i in (25) can be reduced to M line integrals of a constant vector, $(\mu_o / 4\pi r_o)\, dl_i$, over M closed loops. Because the line integral of a constant vector over any closed loop is always zero, these changes, therefore, have no effect on the branch voltage V_i:

$$\begin{aligned} V_i &= \sum_{j=1}^{N} \left(M_{ij} + \Delta M_{ij}\right) \frac{dI_j}{dt} \\ &= \sum_{j=1}^{N} M_{ij} \frac{dI_j}{dt} - \sum_{m=1}^{M} \left\{ \cancelto{=0}{\oint_{l_m} \frac{\mu_o}{4\pi r_o} dl_i \cdot dl_m} \right\} \frac{dI_m}{dt} \end{aligned} \tag{26}$$

It must be emphasized, however, that all mutual inductances must be included in (25) to properly represent the physical system. Without this restriction, (25) represents a system with currents closing at infinity, hence a non-physical one.

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