Delay Analysis of VLSI Interconnections Using the Diffusion Equation Model *

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Abstract

The traditional analysis of signal delay in a transmission line begins with a lossless LC representation, which yields a *wave equation* governing the system response; 2-port parameters are typically derived and the solution is obtained in the transform domain. In this paper, we begin with a distributed RC line model of the interconnect and analytically solve the resulting *diffusion equation* for the voltage response. A new closed form expression for voltage response is obtained by incorporating appropriate boundary conditions for interconnect delay analysis. Calculations of 50% and 90% delay times for various cases of interest (e.g., open-ended RC line) give substantially different estimates from those commonly cited in the literature, thus suggesting revised delay estimation methodologies and intuitions for the design of VLSI interconnects. The discussion furthermore provides a unifying treatment of the past three decades of RC interconnect delay analyses.

1 Overview

Delay analysis of VLSI interconnections is a key element in timing verification, gate-level simulation and performance-driven layout design. The standard approach to modeling interconnect delay has been based on a simple *lossless* LC model which considers only inductances (L) and capacitances (C). For this lossless model, the relationship between v and i gives rise to a second-order partial differential equation of the form [9]:

$$\frac{\partial^2 v}{\partial x^2} = LC \cdot \frac{\partial^2 v}{\partial t^2} \tag{1}$$

and the solution to this wave equation is of the form

$$v(x) = A_1 e^{\theta x} + A_2 e^{-\theta x} \tag{2}$$

where $\theta \equiv$ propagation constant $\equiv \jmath\omega\sqrt{lc}$ (l and c are the inductance and capacitance per unit length, and ω is the wave frequency).

One easily extends this model to *lossy* (RLC) interconnects by incorporating a series resistance. The same equations obtained for the lossless model can be used, with $Z_L = R + \jmath\omega L = \jmath\omega[\frac{R}{\jmath\omega} + L]$, i.e., $Z_L = \jmath\omega L'$ where $L' = \frac{R}{\jmath\omega} + L$ is the new inductance value. Similarly, one may incorporate a conductance G via $Z_C = G + \jmath\omega C = \jmath\omega[\frac{G}{\jmath\omega} + C]$, i.e., $Z_C = \jmath\omega C'$ where $C' = \frac{G}{\jmath\omega} + C$ is the new capacitance. The same solution derived for the lossless model can incorporate the new L' and C' values to capture the attenuation factor in lossy lines.

Using the solution (2) to the wave equation, and the characteristic impedance of the line, one may treat the interconnect line as a 2-port and obtain equations for voltage and current at the terminal side of the 2-port in terms of voltage and current at the source side. This yields the 2-port matrix parameters, e.g., $ABCD$ parameters. To obtain the transient time-domain response of an interconnect line, the standard approach has been to calculate the response in the transform domain using 2-port parameters, and then apply inverse transforms to obtain the response in the time domain. We call this the *LC analysis*, or *wave equation*, approach. Since it may be complicated to apply the inverse transforms, various approximations are typically made which simplify the resulting expressions for the time-domain response.

For the well-studied case of an RC transmission line, the traditional LC analysis is extended to an RLC analysis after which L is set to zero. But by contrast, if we initially model the interconnect as a pure distributed RC line, we obtain a *diffusion equation* (or *heat equation*) from which the solution for the transient response, depending on boundary conditions, can be calculated analytically. This *RC-based* delay analysis approach, and its implications, are the subject of the present paper.

Our motivation for adopting the RC-based delay analysis is as follows. For previous generation interconnects, such as for PCB, the resistance per unit length (r) is considerably smaller than the inductive impedance (ωl), i.e., $r \ll \omega l$, so that the conventional LC-based analysis seems reasonable. However, with small feature sizes of thin-film and IC interconnects, we now find that $r \gg \omega l$ up to frequencies of $O(1)$

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GHz, and even at frequencies above $O(1)$ GHz, both terms are of comparable magnitude [13]. Thus, in the present regime of highly resistive interconnects, it seems natural to begin with an RC, rather than LC, model in obtaining the delay estimate.

While the RC-based perspective and the resulting diffusion equation have been noted by many authors, no closed form expression for the voltage response has been derived using appropriate boundary conditions. Indeed, this is a central contribution of our work. Our analysis based on the diffusion equation yields a simple analytical expression for the voltage response. Furthermore, though the solution of the diffusion equation does not refer directly to any wave propagation mechanism, we may yet consider reflections at discontinuities through the analogy of voltage propagation by electromagnetic vibrations (waves) to the propagation of heat waves [5].

To achieve a comparison with previous works, we study the case of an open-ended RC line with ideal source, as well as other idealized cases which have been treated in the literature. Delay estimates calculated from our diffusion equation analysis are substantially different from previous delay estimates, and we believe that this discrepancy may prove significant for future efforts in interconnect modeling and design. We furthermore extend our delay analysis to the case of arbitrary source and load impedances by considering reflections at the source and load. Finally, our analysis afford delay calculations at arbitrary locations on the distributed RC line.

2 Previous Delay-Time Approximations for an RC Line

2.1 Lumped Models

Approximating the interconnect resistance and capacitance by lumped values R and C gives the time-domain response $v(t) = V_0(1 - e^{-\frac{t}{RC}})$ where V_0 is the input voltage applied. With this model, the delay to reach the 63% voltage threshold is $1.0RC$.

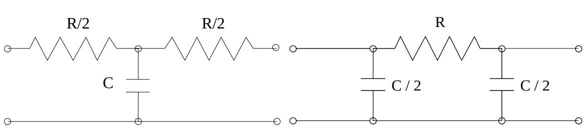

Figure 1: **T** and **Π** elements used in modeling a distributed RC line.

Many works (e.g., [23, 24]) model a distributed RC interconnect using a simple **T** or **Π** configuration, which gives a first-order "lumped-distributed" model or a single-pole response. The transfer function (or Laplace transform of the impulse response) for a **T** or **Π** configuration (Figure 1) is given by

$$H(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{1 + s\frac{RC}{2}}$$

This results in a lumped circuit element with time constant $T = \frac{RC}{2}$.

2.2 Distributed Models

2.2.1 Transient Response Using Laplace Transform

The $ABCD$ parameters of a distributed RC transmission line are [9]:

$$\begin{pmatrix} V_1(s) \\ I_1(s) \end{pmatrix} = \begin{pmatrix} cosh(\theta h) & Z_0 sinh(\theta h) \\ \frac{1}{Z_0} sinh(\theta h) & cosh(\theta h) \end{pmatrix} \begin{pmatrix} V_2(s) \\ I_2(s) \end{pmatrix}$$

where $\theta h = \sqrt{sRC} = \sqrt{j\omega RC}$. V_1 and V_2 corresponds to the voltages at source and load end of the line respectively.

The corresponding open-ended transfer function is

$$H(s) = \frac{1}{\cosh\sqrt{sRC}} \quad (3)$$

Wilnai [25] considers the Laplace transform of the step response with magnitude V_0 for an open-ended distributed RC line,

$$V_2(s) = \frac{V_0}{s\cosh\sqrt{sRC}} \quad (4)$$

with R and C respectively denoting the lumped values of the line resistance and capacitance. Equation (4) is the basis of a number of analyses which are derived from the 2-port model. Using $\cosh x = \frac{e^x + e^{-x}}{2}$ and making the approximation $\cosh x \approx \frac{e^x}{2}$ for $Re\sqrt{sRC} \gg 1$ (i.e., the high-frequency leading edge of the step input), Wilnai obtains the approximate time-domain response

$$v_2(t) \approx 2V_0[1 - erf(\sqrt{\frac{RC}{4t}})] \quad (5)$$

$$v_2(t) \approx V_0[1 - 1.366e^{-\frac{2.5359t}{RC}} + 0.366e^{-\frac{9.4641t}{RC}}] \quad (6)$$

for the cases $t \ll RC$ and $t \gg RC$, respectively. Using Equation (6), Wilnai obtains a value of $1.02RC$ for the 90% delay time and a value of $0.37RC$ for the 50% delay time. By writing

$$\frac{1}{s\cosh\sqrt{sRC}} = \frac{1}{s} \cdot \frac{2}{e^{\sqrt{sRC}}(1 + e^{-2\sqrt{sRC}})} \quad (7)$$

and using $\frac{1}{1+e^{-2\sqrt{sRC}}} = \sum_{n=0}^{\infty}(-e^{-2\sqrt{sRC}})^n$, Mattes [16] has recently obtained a more precise solution of Equation (4) which yields estimates of $1.06RC$ for the 90% delay time, and $0.37RC$ for 50% delay time.

Peirson and Bertnolli [18] have also calculated the transfer function of an open-ended distributed RC line; by using reciprocal time domain analysis they find approximate time domain expressions for the transfer function. From these impulse responses, we may easily derive the system responses for unit step input using Laplace transform tables:

$$v_2(t) \approx \ 2V_0[1 - erf(\sqrt{\frac{RC}{4t}}) \quad (t \ll RC) \quad (8)$$

$$v_2(t) \approx V_0 \tfrac{4\sqrt{RC}}{\pi}[1 - e^{-2.467\frac{t}{RC}}] \quad (t \gg RC) \tag{9}$$

Sakurai [20] uses a similar 2-port model and obtains the voltage response as

$$v_2(t) = V_0(1 - 1.273e^{-2.467\frac{t}{RC}} + 0.424e^{-22.206\frac{t}{RC}}) \tag{10}$$

These works, particularly [25], have had great influence on the literature. For example, Saraswat and Mohammadi [21] use the results of [15, 25] to obtain their rise time estimates. Bakoglu and Meindl [4] also cite Wilnai's derivation, and write: "Under step-voltage excitation, the times (T) required for the output voltage of distributed and lumped RC networks to rise from 0 to 90 percent of their final values are $1.0RC$ and $2.3RC$, respectively." ([4], p. 904). The authors of [4] go on to state a "very good approximation for delay":

$$\begin{aligned} T &= 1.0R_{int}C_{int} + 2.3(R_{tr}C_{int} + R_{tr}C_L + R_{int}C_L) \\ &\approx (2.3R_{tr} + R_{int})C_{int} \end{aligned} \tag{11}$$

(R_{int} and C_{int} are respectively the interconnect resistance and capacitance, R_{tr} is the output resistance of the driving transistor and C_L is the load capacitance). This last expression (11) has been frequently invoked in the literature (see [1], [22] or the book [3]).

Interestingly, the voltage response for a step input using the 2-port model has been rederived many times in the literature. For example, Antinone and Brown [2] express $cosh(\sqrt{sRC})$ as an infinite product series and then consider only the first three terms of the product expansion. This is not a good approximation because the coefficients of s and s^2 are not exact, and depend heavily on the number of terms used in the product expansion. Mey [17] noted the crudeness of this approximation and proposed an infinite partial fraction expansion, thus obtaining the same solution as Sakurai. Ghausi and Kelly [11] are yet another group who earlier published the identical analysis.

The common feature of all these works is that they use the 2-port transfer matrix of the distributed RC line to obtain their respective time-domain estimates of the transient response. The 2-port parameters for the distributed RC line are obtained from the solution of the *wave equation* (2) for v and i (see, e.g., [9]). But as we discuss in Section 3 below, voltage or current in a pure distributed RC line obeys a *diffusion equation*.

2.2.2 A Previous Time-Domain Analysis

Finally, a solution which uses time-domain analysis is that of Kaufman and Garrett [15], who formulate a distributed RC model for interconnect and derive a diffusion equation for voltage on the line. However, to obtain the transient response to a step input, [15] makes the simplifying assumption $v(x,t) = f(x) \cdot g(t)$, namely, separability of the voltage response into separate functions of time and position; this leads to a complicated and special-case solution

$$v_2(t) = 1 - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}e^{-((2n+1)\pi/2)^2 t/RC} \tag{12}$$

Considering only the first few terms of the series yields

$$v_2(t) \approx (1 - 1.273e^{-2.467\frac{t}{RC}} + 0.424e^{-22.206\frac{t}{RC}}) \tag{13}$$

This expression is different from that of Wilnai or Peirson, but is identical[1] to that given by Sakurai (Equation 10). The book of Ghausi and Kelly [11] gives an analysis using the same separability assumption. These previous delay approximations are summarized in Table 1 below.

3 The Diffusion Equation Analysis

3.1 Obtaining the Diffusion Equation from the Distributed RC Model

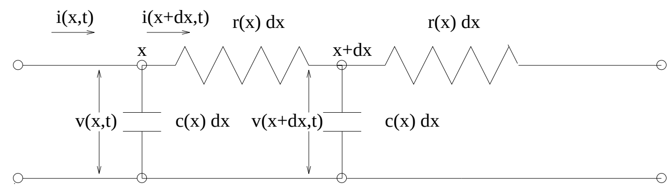

Figure 2: Lumped approximation for Δx in a distributed RC line.

The diffusion equation for voltage in a distributed RC line can be derived from first principles as follows (see [15]). Consider a lumped approximation for Δx of the line, as shown in Figure 2. By applying simple nodal equations at the nodes x and $x + \Delta x$, we obtain

$$\begin{aligned} i(x,t) &= c(x)\Delta x\frac{\partial v(x,t)}{\partial t} + i(x+\Delta x, t) \\ v(x,t) &= r(x)\Delta x i(x+\Delta x,t) + v(x+\Delta x,t) \end{aligned}$$

where $r(x)$ and $c(x)$ are resistance and capacitance per unit length. As $\Delta x \to 0$, and for constant $r(x)$ and $c(x)$, the above equations reduce to the **diffusion equation**

$$rc\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}. \tag{14}$$

The solution of Equation (14) can be obtained by restricting it to the set of solutions of the form $v(\frac{x}{\sqrt{t}})$ using the substitution variable $\eta = x\sqrt{\frac{rc}{2t}}$ [14]. This is the appropriate substitution for a parabolic equation. We obtain

$$v(\eta) = C_1\int_0^{\eta} e^{-\frac{\eta^2}{2}}d\eta + C_2 \tag{15}$$

One can also obtain this solution directly from the heat kernel for the diffusion equation [7]. This should be contrasted with the solution of [15], which must assume the separable form for $v(x,t)$. Of course, the solution we obtain will be highly dependent on the boundary conditions that apply; in particular, we are interested in the well-studied case of the open-ended distributed RC line.

[1] In the taxonomy that we present below, the result of Kaufman and Garrett is characterized as "approximate" because their method must assume the special form of the diffusion equation solution. On the other hand, Sakurai's result is "exact" with respect to the distributed RC 2-port analysis.

Method	Accuracy/ Regime	Time-Domain Voltage Response
Simple Lumped Model	Approximate	$V_0(1-e^{-\frac{t}{RC}})$
Wilnai's 2-port Model	Small t Large t	$2V_0(1-erf(\sqrt{\frac{RC}{4t}}))$ $V_0(1-1.366e^{-2.5359\frac{t}{RC}}+0.366e^{-9.4641\frac{t}{RC}})$
Mattes's 2-port Model	Exact	$2V_0\sum_{n=1}^{\infty}(-1)^{n-1}\left(1-erf(\frac{2n-1}{2}\sqrt{\frac{RC}{t}})\right)$
Peirson's 2-port Model*	Small t Large t	$2V_0(1-erf(\sqrt{\frac{RC}{4t}}))$ $V_0 1.273\sqrt{RC}(1-e^{-2.467\frac{t}{RC}})$
Sakurai's 2-port Model	Exact but approximated to three terms	$V_0(1-1.273e^{-2.467\frac{t}{RC}}+0.424e^{-22.206\frac{t}{RC}})$
Kaufman's Diffusion Equation Model	Heuristic derivation but approximated to three terms	$V_0(1-1.273e^{-2.467\frac{t}{RC}}+0.424e^{-22.206\frac{t}{RC}})$
Antinone/Brown's 2-port model	Approximate	$V_0(1-1.172e^{-2.467\frac{t}{RC}}+0.195e^{-22.206\frac{t}{RC}}-0.023e^{-61.685\frac{t}{RC}})$
Our Diffusion Equation Model	Exact	$V_0(1-erf(\sqrt{\frac{RC}{4t}}))$

Table 1: Voltage response of an open-ended distributed RC line under a step input excitation of magnitude V_0. $(*)$ Response calculated from the transfer function in [PB69].

3.2 Boundary Conditions

For the distributed RC line, we derive the two boundary conditions necessary to solve Equation (15) as follows.

Boundary Condition 1: At $t=0$, the line is quiet and $v(x,t)=0$ for all x, i.e.,

$$C_1\sqrt{\frac{\pi}{2}}+C_2=0.$$

Therefore,

$$C_2=-C_1\sqrt{\frac{\pi}{2}} \tag{16}$$

Note that this boundary condition applies to every new wave that is born due to reflection.

Boundary Condition 2: The second boundary condition is obtained from the structure of the input applied at the front end of the transmission line ($x=0$) and in terms of the rise-time value, t_{rise}. Notice that the voltage at $x=0$ depends on the source impedance, Z_S, and the characteristic impedance of the line, Z_0, since this structure acts as a voltage divider. Therefore the voltage at the beginning of the line, i.e at $x=0$, in the transform domain is given by

$$V_1(s)=(\frac{Z_{in}}{Z_{in}+Z_S})\frac{V_0}{s} \tag{17}$$

where Z_{in} is the input impedance looking into the interconnect line.

At a given rise-time ($t=t_{rise}$), the voltage $V_1(0,t_{rise})$ at the front end of the transmission line can be obtained from the time domain representation of Equation (17). Let $\alpha_{rise}V_0$ be this voltage at the rise-time, i.e., $V_1(0,t_{rise})=\alpha_{rise}V_0$, where $0<\alpha_{rise}\leq 1$.

Substituting into Equation (15) and evaluating at $x=\epsilon$, with ϵ tending to 0, we obtain

$$\alpha_{rise}V_0=C_1\int_0^{\epsilon\sqrt{\frac{rc}{2t_{rise}}}}e^{(-\frac{\eta^2}{2})}d\eta+C_2 \tag{18}$$

3.3 Solution of the Diffusion Equation

We use (16) and (18) to solve for C_1 and C_2 :

$$\alpha_{rise}V_0=C_1\int_0^{\epsilon\sqrt{\frac{rc}{2t_{rise}}}}e^{(-\frac{x^2}{2})}dx-C_1\sqrt{\frac{\pi}{2}}$$

yields

$$C_1=\alpha_{rise}V_0\cdot\frac{1}{[\int_0^{\epsilon\sqrt{\frac{rc}{2t_{rise}}}}e^{-\frac{x^2}{2}}dx-\sqrt{\frac{\pi}{2}}]}$$

from which

$$V(\eta)=\kappa\alpha_{rise}V_0[1-erf(\frac{\eta}{\sqrt{2}})] \tag{19}$$

where $\kappa = \frac{1}{[1-erf(\frac{\epsilon}{2}\sqrt{\frac{rc}{t_{rise}}})]}$.

To achieve the case of an ideal step input, the boundary condition at $x = 0$ should be evaluated for t_{rise} tending to zero and α_{rise} tending to 1, i.e., we let $t_{rise} = \varepsilon$ with $\varepsilon \to 0$. Then, the error function argument in the expression for κ will tend to zero, since ϵ and ε both tend to zero with the numerator of higher degree than the denominator, i.e., $\lim_{\epsilon\to 0,\varepsilon\to 0} \frac{\epsilon}{2}\sqrt{\frac{rc}{\varepsilon}} = 0$. Therefore, $\kappa = 1$ and the diffusion equation solution reduces to

$$V(\eta) = V_0[1 - erf(\frac{\eta}{\sqrt{2}})]. \qquad (20)$$

Observe that the same result is obtained when the input corresponds to an ideal source, i.e., $Z_S = 0$, $V_1(s) = \frac{V_0}{s}$, with voltage at $x = 0$ constant for all t and equal to V_0. In this case, $V(0,t) = V_0 u(t)$ and using this condition in Equation (15) yields

$$V_0 = C_1 \int_0^{\eta = x\sqrt{\frac{rc}{2t}}=0} e^{-\frac{x^2}{2}} dx + C_2$$

from which $C_2 = V_0$ and

$$V(\eta) = V_0[1 - erf(\frac{\eta}{\sqrt{2}})]. \qquad (21)$$

This result is the same as Equation (20), as we expect.

The same Equation (20) can also be obtained by using the Boundary Condition 2 and another boundary condition which captures the open end of the line [14]. We believe that the voltage on the line will better obey Equation (20) near the source than near the load; ongoing experimental efforts are aimed at validating this belief. A comment is in order: the two boundary conditions we use are discontinuous (at $x = 0, t = 0$), but this discontinuity smooths immediately and the solution is still valid.

3.4 Threshold Delay Calculations Using Diffusion Equation Model

We now proceed with delay calculations for a case that has been of interest throughout the literature, namely, the open-ended distributed RC line with an ideal source. Recall that with an ideal step input, $\kappa = 1$ in (19), the solution reduces to that given in (20). The equation for the voltage response for an ideal source is obtained in Equation (21). Using error function tables, we easily calculate the time for a signal applied at the input of an interconnect to reach a given threshold voltage at distance x (on the line) from the input terminal. For example, our diffusion equation solution implies $2.18 R_x C_x$ for 63% delay time.

One can see that the delay times for the diffusion equation model are substantially different from those commonly employed in the literature, i.e., the Elmore delay (for a single RC line) of $t_x(63\%) = \frac{R_x C_x}{2}$. This difference would certainly affect standard delay estimates: for example, minimum clock skew routing results which use lumped models will be affected when modified to consider the new delay values obtained above. We emphasize that our result does not imply that the previous LC-based approach is wrong. Rather, our work simply shows that solving the diffusion equation for RC interconnects yields a very different perspective on delay calculations. A careful experimental investigation is needed to determine the respective regimes for which these models are valid.

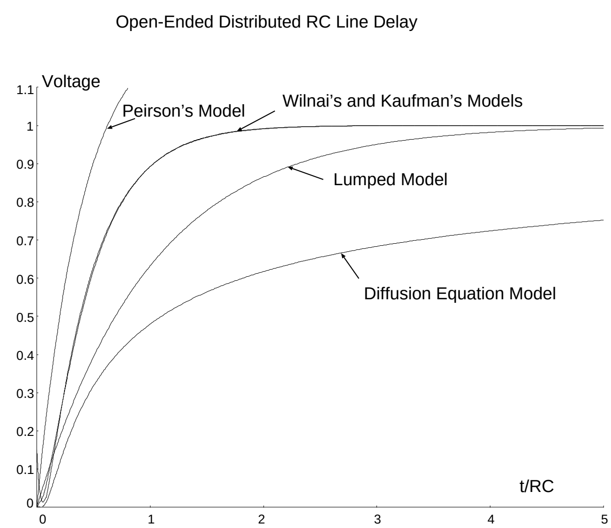

Figure 3: Unit step response for lumped RC model and various distributed RC models. Note that Wilnai's and Kaufman's models are not identical: Kaufman's ($\equiv$ Sakurai's) model gives a non-monotone response. Also note that the response in Peirson et al.'s model is dependent on the RC constant; we have plotted the response for $RC = 1$.

4 Extensions of the Diffusion Equation Analysis

We close our development by extending the basic result of Equation (20) in two ways: (i) introducing an analysis of reflections, and (ii) incorporating nonzero time of flight into the analysis.

4.1 Analysis of Reflections

Recall that the total voltage on the line is given by the summation of the incident wave and all reflected wave components. The reflections are due to discontinuities, e.g., at the source (S) and load (L). In other words,

$$\vec{V}_{Tot}(\eta) = V_I(\eta) + \sum_{i=1}^{\infty} \vec{V}_{R_i}(\eta) \qquad (22)$$

where $V_I(\eta) \equiv$ voltage corresponding to the incident wave and $\vec{V}_{R_i}(\eta) \equiv$ voltage corresponding to the i^{th} reflected wave. Neglecting the time of flight, at any time the expressions for any individual $\vec{V}_{R_i}(\eta)$ will be of the same form as the incident voltage expression $V_I(\eta)$, but with different initial voltage. The solution for $V_I(\eta)$ is given by (19) in general, and by (20) for an ideal step input. We use these expressions to calculate the total voltage on the line for general source and load impedances. Note that since Z_S and Z_L are in general complex, we must treat $V_{Tot}(\eta)$ and $V_{R_i}(\eta)$ as phasors $\vec{V}_{Tot}(\eta)$ and $\vec{V}_{R_i}(\eta)$ which are functions of time (t), distance (x) and frequency (ω). If $\vec{V}_{Tot}(\eta)$ is a phasor, the delay calculations should compare the magnitude of total voltage (i.e., absolute value of the phasor, $|\vec{V}_{Tot}(\eta)|$) with the threshold value of interest. Luckily, the assumptions in typical cases of interest, while in some ways unrealistic, make the analysis tractable and afford lower bounds for the delay estimates.

In the general case, the reflection coefficients are $\Gamma_S = \frac{Z_S - Z_0}{Z_S + Z_0}$ and $\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$. The first reflection at the load yields

$$\vec{V}_{R_1}(\eta) = \Gamma_L V_I(\eta);$$

the second reflection at the source yields

$$\vec{V}_{R_2}(\eta) = \Gamma_S \Gamma_L V_I(\eta);$$

and in general the i^{th} reflection gives $\vec{V}_{R_i}(\eta) = \Gamma_L^{\frac{i}{2}} \Gamma_S^{\frac{i}{2}} V_I(\eta)$ for i even, with the case of i odd being analogous. Using

$$\begin{aligned} \vec{V}_{Tot}(\eta) &= V_I(\eta) + \sum_{i=1}^{\infty} \vec{V}_{R_i}(\eta) \\ &= V_I(\eta)[1 + \Gamma_L + \Gamma_S\Gamma_L + \ldots + \Gamma_L^{\frac{i}{2}}\Gamma_S^{\frac{i}{2}} + \ldots] \end{aligned}$$

and separating odd and even terms of the summation, we obtain the general solution

$$\vec{V}_{Tot}(\eta) = \kappa\alpha_{rise}(\frac{1+\Gamma_L}{1-\Gamma_L\Gamma_S})V_0[1 - erf(\frac{\eta}{\sqrt{2}})]$$

The form of this expression, i.e., as a sum of error function terms, is quite intuitive.

We may again consider examples of source and load impedances which have received particular attention throughout the literature.[2]

Case 1: Finite-length, open-ended RC transmission line with ideal source.

With $Z_L = \infty$ (i.e., $\Gamma_L = 1$), an ideal source ($Z_S = 0$, $\Gamma_S = -1$ and $\alpha_{rise} = 1$) implies that all of the input voltage appears at $x = 0$. Recalling that $\vec{V}_{Tot}(\eta) = V_I(\eta) + \sum_{i=1}^{\infty} \vec{V}_{R_i}(\eta)$, we see that with reflection coefficients equal to $+1$ or -1, all reflections will cancel. Thus, the summation term in the equation disappears and we obtain the same result as in Equation (21):

$$\vec{V}_{Tot}(\eta) = V_I(\eta) = V_0[1 - erf(\frac{\eta}{\sqrt{2}})]$$

[2] Note that Γ_L and Γ_S will always have an implicit frequency dependence, and therefore $|\vec{V}_{Tot}(\eta)|$ will also depend on the frequency.

Case 2: Finite-length, open-ended RC transmission line with perfectly matched source.

With $Z_L = \infty$ (i.e., $\Gamma_L = 1$) and source impedance $Z_S = Z_0$ (i.e., $\Gamma_S = 0$), there is only the single (initial) reflection at the load. Thus,

$$\begin{aligned} \vec{V}_{Tot}(\eta) &= V_I(\eta) + \vec{V}_{R_1}(\eta) \\ &= 2[\kappa\alpha_{rise}V_0(1 - erf(\frac{\eta}{\sqrt{2}}))] \end{aligned}$$

In practice, the open-ended approximation of an interconnect is often used since the input impedance of MOS devices is typically high compared to the characteristic impedance of the line.

4.2 Non-Zero Time of Flight

Last, we note that the derivation of Equation (22) neglected the time of flight, T_{fl}, and used identical expressions for incident and reflected waves. For a maximum on-chip interconnect of length approximately $1cm$, the time of flight will be around 0.1ns [6]. However, the length of a typical single interconnect segment will be much smaller (on the order of 0.01cm) and T_{fl} will be on the order of picoseconds. Since delays for typical operating frequencies of $O(1)$ GHz are of the order of 0.1ns [8], one may reasonably neglect T_{fl} in delay calculations, as has usually the case in the literature. Nevertheless, T_{fl} becomes significant with shorter rise or delay times, or with longer interconnects (e.g., for large die or MCM substrates). Here, we show that our analysis can extend to non-zero T_{fl}. The key observation is that taking T_{fl} into account will only *increase* our delay estimates, and these estimates are already larger than those in the literature.

To account for a non-zero time of flight, we simply record an additional displacement of T_{fl} for each successive reflection. The total voltage on the line after the i^{th} reflection is

$$\vec{V}_{Tot}(x,t) = V_I(x\sqrt{\frac{rc}{2t}}) + \sum_{i=1}^{\infty} \vec{V}_{R_i}(x\sqrt{\frac{rc}{2(t - iT_{fl})}})$$

For example, if we consider non-zero T_{fl} in **Case 2** of the previous subsection, the 90% delay time for a line of length h is obtained as follows.

$$V_I(h\sqrt{\frac{rc}{2t}}) + \vec{V}_{R_1}(h\sqrt{\frac{rc}{2(t - T_{fl})}}) = 0.9V_0$$

which implies

$$erf(\frac{h}{2}\sqrt{\frac{rc}{t_h}}) + erf(\frac{h}{2}\sqrt{\frac{rc}{t_h - T_{fl}}}) \approx 1.1$$

As expected, non-zero T_{fl} will increase the 90% delay time. The **Case 2** analysis provides an upper bound on the voltage response, so the actual delay time is lower-bounded by this equation. One could solve the equation through an iterative process, starting from a value that is computed assuming $T_{fl} = 0$. For other cases, e.g., **Case 1** above, $T_{fl} > 0$ does not affect the previous analysis since there are no reflections.

5 Summary

A survey of three decades of interconnect delay analyses reveals that the analysis of signal delay in a transmission line is traditionally performed starting with a lossless *LC* representation and a *wave equation* for the system response; the solution is obtained in the transform domain via 2-port parameters. In this paper, we begin with a distributed *RC* line model of the interconnect, which yields a *diffusion equation* for the voltage response. We have given a new analytic solution of this equation incorporating appropriate boundary conditions, and have obtained estimates for 50% and 90% delay times at arbitrary locations on the interconnect line that differ substantially from the delay estimates currently employed in the literature. Beyond its many implications for revised delay estimation methodologies (e.g., for performance-driven routing tree construction, minimum-skew clock distribution, or buffer placement), our time-domain solution also yields new intuitions regarding design objectives for VLSI interconnects. Our approach also handles the case of non-zero T_{fl}.

Our result does not imply that the previous *LC*-based approach is wrong, but rather shows that solving the diffusion equation for *RC* interconnects can provide a totally new perspective on delay calculations. Thus, we are also pursuing the central challenge of experimentally validating both our new model and previous delay models in various technology regimes.

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