Evolutionary Algorithms for Two Problems from the Calculus of Variations

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Abstract. A brachistochrone is the path along which a weighted particle falls most quickly from one point to another, and a catenary is the smooth curve connecting two points whose surface of revolution has minimum area. Two evolutionary algorithms find piecewise linear curves that closely approximate brachistochrones and catenaries.

Two classic problems in the calculus of variations seek a brachistochrone, the path along which a weighted particle falls most quickly from one point to another, and a catenary, the smooth curve of arc length l between two points whose surface of revolution has minimum area. Analytical solutions to these problems have long been known. In a uniform gravitational field and without friction, a brachistochrone is an arc of a cycloid, the curve traced by a point on a rolling circle. The curve of specified length whose surface of revolution has minimum area is a catenary, an arc of a hyperbolic cosine.

Two evolutionary algorithms seek approximate solutions to these problems. They search spaces of piecewise linear functions, which they represent as sequences of y-coordinates associated with evenly-spaced x's. The algorithms apply mutation and $(\mu+\mu)$ reproduction to the chromosomes in their populations. The mutation operator perturbs elements of parent chromosomes with random values from a normal distribution with mean zero, and the standard deviation of this distribution diminishes as each algorithm runs.

Previously, Zitar and Homaifar [2] described a genetic algorithm for the brachistochrone problem, and Erickson, Killmer, and Lechner [1] applied genetic programming to it.

In the EA for the brachistochrone problem, a chromosome's fitness, which the EA seeks to minimize, is the time required for the particle to fall along the path that the chromosome represents. Consider the particle as it passes through a point $P = (x, y)$ on a curve. The particle was initially at rest, and its potential energy relative to P was mgy, where m is the particle's mass and g is the acceleration due to gravity. At P, the particle is moving with velocity v and its kinetic energy is $\frac{1}{2}mv^2$. Energy is conserved, so $mgy = \frac{1}{2}mv^2$. Thus the particle's velocity at P is $v = \sqrt{2gy}$.

The length of the segment from (x_i, y_i) to (x_{i+1}, y_{i+1}) is $d_i = \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}$, and the particle's average velocity over the segment is the average of its velocities at the segment's endpoints: $v_i = (\sqrt{2gy_i} +$

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$\sqrt{2gy_{i+1}})/2$. The time the particle spends traversing the segment is then $t_i = \frac{d_i}{v_i}$, and the time that the particle requires to fall the entire distance on the path a chromosome specifies is the sum of these values over all the subintervals.

In the EA for the catenary problem, we seek to minimize both the area of the surface generated by rotating the represented curve about the x-axis and the difference between the length of that curve and a target length l. The segment connecting (x_i, y_i) and (x_{i+1}, y_{i+1}) sweeps out a frustum of a cone whose surface area is $A_i = 2\pi r_i d_i$, where $r_i = (y_i + y_{i+1})/2$ is the average radius of the frustum and d_i is again the length of the segment. The area of the surface is the sum of the areas of these frusta, and the length of the curve is the sum of the lengths of its segments. A chromosome's fitness is a weighted sum of the former and the deviation of the latter from l:

$$\text{fitness} = w_A \cdot \text{area} + w_d \cdot |\text{length} - l| \ .$$

In tests of the two algorithms, the number of linear segments was $n = 30$. The EAs' populations contained $\mu = 50$ chromosomes, each of $n+1 = 31$ values. The initial standard deviation of the mutating distribution was 1.0, and it was multiplied by 0.998 after each generation. Each algorithm ran through 2 500 generations. In the EA for the catenary problem, $w_A = 1$ and $w_d = 25$.

Both EAs were effective on a variety of test instances. Figure 1(a) shows a curve of rapid descent generated by the first algorithm; it is indistinguishable from the optimum arc of a cycloid. Figure 1(b) shows a curve of small area and specified length generated by the second algorithm; it is indistinguishable from the optimum arc of a hyperbolic cosine.

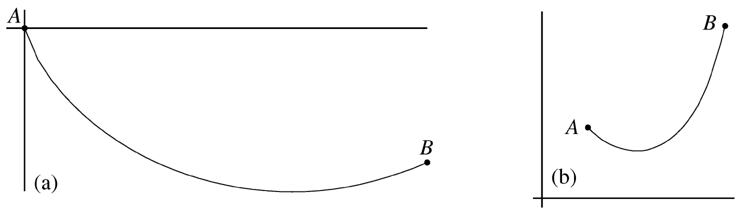

Fig. 1. (a) An approximate brachistochrone generated by the first EA and (b) an approximate catenary generated by the second EA

References

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