Evolution Strategies with Exclusion-Based Selection Operators and a Fourier Series Auxiliary Function

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Abstract. To improve the efficiency of the currently known evolutionary algorithms, we have proposed two complementary efficiency speed-up strategies in our previous research work respectively: the exclusion-based selection operators and the Fourier series auxiliary function. In this paper, we combine these two strategies together to search the global optima in parallel, one for optima in large attraction basins and the other for optima in very narrow attraction basins respectively. They can compliment each other to improve evolutionary algorithms (EAs) on efficiency and safety. In a case study, the two strategies have been incorporated into evolution strategies (ES), yielding a new type of accelerated exclusion and Fourier series auxiliary function ES: the EFES. The EFES is experimentally tested with a test suite containing 10 complex multimodal function optimization problems and compared against the standard ES (SES). The experiments all demonstrate that the EFES consistently and significantly outperforms the SES in efficiency and solution quality.

1 Introduction

Evolutionary algorithms (EAs) are global search procedures based on the evolution of a set of solutions viewed as a population of interacting individuals. They have been successfully used for optimization problems. But for solving large scale and complex optimization problems, EAs have not demonstrated themselves to be very efficient [4] [5]. We believe the main factor which causes low efficiency of the current EAs is the convergence towards undesired attractors. This phenomenon occurs when the objective function has some local optima with large attraction basins or its global optimum is located in a small attraction basin in a minimization case. The relationship between the convergence to a global minimum and the geometry (landscape) of the difficult function problems is very important. If the population of EAs gets trapped into suboptimal states, which locate in comparative large attraction basins, then it is difficult for the variation operators to produce an offspring which outperforms its parents. In the second case, if global optima are located in relatively small attraction basins, and the individuals of EAs have not found these basins yet, the probability of the variation operators to produce offspring which locate in these small attraction basins

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is quite low. In both cases, the stochastic mechanism of EAs yields unavoidable resampling, which increases the algorithm's complexity and decelerates the search efficiency.

To overcome these two limitations that cause low efficiency of the currently known EAs, we have proposed two complementary efficiency speed-up strategies in our previous research work respectively [6]: the exclusion-based selection operators and the Fourier series auxiliary function. The exclusion-based selection operators could efficiently prevent the individuals of EAs from getting into the attraction basins of local optima. While the Fourier series auxiliary function could guide an algorithm to search for optima with small attraction basins efficiently. Moreover, this strategy can compensate the deficiency of the exclusion-based selection operators on the algorithm's safety, i.e. the avoidance of excluding a global optimum contained in a narrow attraction basin. In this paper, we developed a new algorithm, the EFES, which incorporate the exclusion and Fourier series auxiliary function into the evolution strategies (ES) [2]. We expect that EFES will have the advantages of both two strategies — efficiency and safety.

This paper is organized as follows: we will explain the two novel efficiency speed-up strategies for EAs implementation in Sections 2 and 3 respectively. Particularly, a set of "exclusion-based" selection operators is proposed in Section 2 to simulate the "survival of the fittest" principle more powerfully and a Fourier series auxiliary function is introduced in Section 3. In Section 4, we will demonstrate how to embed the exclusion-based selection operators and the Fourier series auxiliary function into ES to generate EFES. In Section 5, the EFES is experimentally examined, analyzed and compared with a set of typical, multi-modal function optimization problems. The last section is the conclusion.

2 Exclusion-Based Selection Operators

This section defines and explores a somewhat different selection mechanism — the exclusion-based selection operators.

Any EAs solve an optimization problem, say,

$$(\mathbf{P}) \qquad min\{f(x) : x \in \Omega\},$$

where $f : \Omega \subset R^n \rightarrow R$ is a function. For simplicity, consider the problem $(\mathbf{P})$ with the domain Ω specified by $\Omega = [u_1, v_1] * [u_2, v_2] * \cdots [u_n, v_n]$. The new scheme is based on the cellular partition methodology. Given an integer d, let $h_i = \frac{v_i - u_i}{d}$, we define

$$\sigma(j_1, j_2, \cdots, j_n) =$$

$$\{x = (x_1, x_2, \cdots, x_n) \in \Omega : (j_i - 1) \times h_i \leq x_i - u_i \leq j_i \times h_i, 1 \leq j_i \leq d\}$$

and let the subregion $\sigma(j_1, j_2, \cdots, j_n)$, be a cell, and the cell collection $\Gamma(d)$ is called a cellular partition of Ω. The vector representing the center of the cell is defined by

$$m(\sigma(j_1, \cdots, j_n)) = ((u_1 + (j_1 + \frac{1}{2}) \times h_1), \cdots, (u_n + (j_n + \frac{1}{2}) \times h_n)).$$

Let us first formalize the exclusion operators with reference to the cellular partition $\Gamma(d)$ of Ω.

Definition 1: *An exclusion operator is a computationally verifiable test for nonexistence of solution to problem* $(\mathbf{P})$ *which can be implemented on every cell of* $\Gamma(d)$.

For example, assume that f satisfies the Lipschitz condition: $|f(x)-f(y)| \leq \alpha||x-y||$. Then, for any $\sigma \in \Gamma(d)$, there is a global minimizer x^* in σ only if $| f(m(\sigma)) - f(x^*) | \leq \alpha||m(\sigma) - x^*|| \leq \frac{\alpha}{2}\omega(\sigma)$, where $\omega(\sigma)$ is the *mesh size* of $\Gamma(d)$, defined by

$$\omega(\sigma) = max_{1\leq i\leq n}\{\frac{v_i - u_i}{d}\}$$

This implies that $f(m(\sigma)) - \frac{\alpha}{2}\omega(\sigma) \leq f(x^*)$ is a necessary condition for the existence of the global optimum of $(\mathbf{P})$. Thus $f(m(\sigma)) - \frac{\alpha}{2}\omega(\sigma) > f(x^*)$ gives a nonexistence test for the solution. This test is an exclusion operator because it can be computationally verified in each cells of $\Gamma(d)$.

By Definition 1, any exclusion operator can serve to computationally test the nonexistence of the solution of $(\mathbf{P})$ in any given cells of $\Gamma(d)$. It therefore can be used to check if a given cell in $\Gamma(d)$ is prospective or not as a global optimum (or, as a portion of a attraction basin of a global optimum) of $(\mathbf{P})$. Accordingly, the non-global optimum, or, more loosely, less prospective cells can be identified and can be deleted from further consideration. This is the key mechanism that is adopted in the present work to accelerate EAs. Hereafter any selection mechanism based on such exclusion principle will be called an *exclusion-based selection* operator.

Let σ be an arbitrary cell in $\Gamma(d)$ with its center $m(\sigma)$ and the mesh size $\omega(\sigma)$. The following provides us with a series of exclusion operators for minimization problems. Please note that these operators (tests) can be equally applied to maximization problems by reversing the inequality signs.

Example 1: *Concavity Test Operator*

Suppose f is continuous, then a necessary condition for a point x to be a global minimizer is that f is convex at the neighborhood of x. Therefore, when d is sufficiently large, the following tests $(\mathbf{E}_1)$ and $(\mathbf{E}_2)$ are exclusion operators.

$(\mathbf{E}_1)$: *There is a direction* $e \in R^n$ *such that*

$$f(m(\sigma)) > \frac{1}{2}[f(x^+) + f(x^-)]$$

where $x^- = x - \eta e, x^+ = x + \eta e$ *and* $0 < \eta \leq \frac{1}{2}\omega(\sigma)$

$(\mathbf{E}_2)$: *There is a direction* $e \in R^n$ *such that* $\max\{f(x^+), f(x^-)\} > f_{best}$ *and*

$$[f(x^+) - f(m(\sigma))][f(x^-) - f(m(\sigma))] < 0$$

where x^+ *and* x^- *are the same as in* (E_1), *and* f_{best} *is the current known best fitness value.*

The tests $(\mathbf{E}_1)$ and $(\mathbf{E}_2)$ can immediately follow from the observations that every $m(\sigma)$ is in the interior of Ω, the $(\mathbf{E}_1)$ features the concave property of f

on cell σ, and ($\mathbf{E}_2$) characterizes the convex and concave overlapped property in which no global optimum of f exists.

Example 2: *Lipschitz Test Operator*

Let $\pounds(f,\alpha)$ denote the family of all continuous functions that satisfies the Lipschitz conditions: $\mid f(x) - f(y) \mid \leq \alpha(\Omega)||x-y||, x, y \in \Omega \in \Gamma(d)$. Then the following tests ($\mathbf{E}_3$) and ($\mathbf{E}_4$) are exclusion operators.

($\mathbf{E}_3$): $f_{best} < f(m(\sigma)) - \frac{\alpha(\sigma)}{2}\omega(\sigma)$, *if* $f \in \pounds(f,\alpha)$

($\mathbf{E}_4$): $f_{best} < f(m(\sigma)) - \frac{\alpha(\sigma)}{8}\omega^2(\sigma)$, if $f' \in \pounds(f,\alpha)$

where f_{best} is again the "best-so-far" fitness value, and f' *is the derivative of* f.

Example 3: *Formal Series Test Operator*

Let $\Im$ be the class of all functions that can be expressed as a finite number of superpositions of formal series and their absolute values $A(f)$ of f is defined by

$$A(f) = A(f^{(1)}) + \sum_{j=2}^{k} A(f^{(j)})(x).$$

For any $f \in \Im$ and $g \succ A(f)$, it is known [8] that the following basic inequality holds:

$$|f(x) - f(y)| \leq A(g)(|y| + |x-y|) - A(g)(|y|), \forall x, y \in R^n$$

This implies, similar to test ($\mathbf{E}_3$), that the following test ($\mathbf{E}_5$) is an exclusion operator.

($\mathbf{E}_5$): *There is a formal series* $g \succ A(f)$*such that*

$$f_{best} \leq f(m(\sigma)) - [A(g)(|m(\sigma)| + \frac{1}{2}\omega(\sigma)) - A(g)(|m(\sigma)|)]$$

Various other exclusion operators may also be constructed by virtue of other delicate mathematical tools such as interval arithmetic [1] and the cell mapping methods [7] [8].

Remark 1:

(i) The exclusion-based selection is the main component and contributor to the accelerated evolutionary algorithms (the fast-EAs). Aiming at suppressing the resampling effect of EAs, such type of selection provides a smart guidance to EAs search towards promising areas through eliminating non-promising areas. Different in principle from the conventional selection operators, the construction of which is based on sufficient condition for the existence of solution to ($\mathbf{P}$), the exclusion-based selection operators can be constructed based on necessary condition. This presents a general methodology of constructing exclusion operators. The above listed tests ($\mathbf{E}_1$)-($\mathbf{E}_5$) show such examples of the construction.

(ii) The application of exclusion-based operator has another advantage: It can very naturally incorporate some useful properties of the objective function into the EAs search, providing other acceleration means whenever possible. Indeed, Examples 1-3 all have taken advantage of properties of f in certain ways (say, continuity in Examples 1, Lipschitz conditions in Example 2 and analyticity in

Example 3). As a general rule, the more exclusive properties of f are utilized, the more accurate a test could be deduced (For example, ($\mathbf{E}_3$) is more accurate than ($\mathbf{E}_4$) when the Lipschitz condition was applied for the derivative f' instead of for f). These different tests deduced from different properties by no means have to be applied for a problem in the same time. They can, for instance, be applied either independently, or with several others together, or totally simultaneously, depending on the available information on f that can actually be made use of.

3 A Fourier Series Auxiliary Function

The exclusion-based selection operators are the efficient accelerating operators based on the interval arithmetic and the cell mapping methods. However, if the function optimization problem is very complex, say there are many sharp basins, the optima may be excluded by mistake using the above operators. Meanwhile, searching an optimum with a small attraction basin is difficult for standard evolutionary algorithms. To solve the above problem, a Fourier series auxiliary function is introduced in this section.

As we know, if $f(x)$ is continuous or merely piecewise continuous (continuous except for finitely many finite jumps in the interval of integration), then the Fourier series of $f(x)$ is convergent. Its sum is $f(x)$, except at a point x_0 at which $f(x)$ is discontinuous and the sum of the series is the average of the left- and right-hand limits of $f(x)$ at x_0 [3]. We define the finite partial sum of the Fourier series called $F_\ell^k(x)$(e.g. for one dimension),

$$F_\ell^k(x) = \sum_{n=\ell}^{k}(a_n \cos \frac{2n\pi}{u-v}x + b_n \sin \frac{2n\pi}{u-v}x), x \in [u, v].$$

The infinite Fourier series $F_1^\infty(x)$ converges to $f(x)$ at any point, but the convergent speed of the finite partial sum $F_1^\ell(x)(\ell < \infty)$ is different at each point. For numerical function optimization, the finite partial sum $F_1^\ell(x^*)(\ell < \infty)$ converges to $f(x^*)$ much slower for optimum x^* with small attraction basin in $f(x)$ than that of optimum x^* with a large attraction basin. This indicates the partial sum $|F_\ell^\infty(x^*)| = |f(x^*) - F_1^\ell(x^*)|$ at x^* with a small attraction basin is larger than at x^* with a large attraction basin. Because when an integer $k \to \infty$, the coefficients of the k^{th} Fourier series term a_k and $b_k \to 0$, the infinite partial sum $F_k^\infty(x) \to 0$. So we consider the finite partial sum $F_\ell^k(x)(\ell < k)$ instead of $F_\ell^\infty(x)$. The proposition $|F_\ell^\infty(x^*)| > |F_\ell^\infty(x)|$ equals to $|F_\ell^k(x^*)| > |F_\ell^k(x)|(\ell < k)$ when x^* locates in a small attraction basin.

The features of the finite partial sum $F_\ell^k(x)(\ell < k)$ include enlarging small attraction basins, and smoothing large attraction basins of $f(x)$ as shown in Fig.1. We have designed three strategies: the region partition strategy, the one-element strategy and the double integral strategy to construct auxiliary function $g(x)$. The first strategy is designed for representing all optima with small attraction basins by the $F_\ell^k(x)(\ell < k)$ with a small number of terms. The second one is for significantly reducing the computational complexity. The last one is

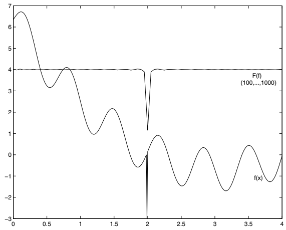

Fig. 1. A schematic illustration for the feature of the Fourier finite partial sum $F_\ell^k(\ell = 100,\ k = 1000)$.

for expanding the dummy optima out of the original feasible region, and keeping the original position of the optimum unchanged (Fig.2). Consequently, we could construct the auxiliary function $g(x)$ using the finite partial sum of the one element of the Fourier trigonometric system

$$g(x) = \sum_{m=\ell}^{k} a_m \prod_{i=1}^{n} \cos mx_i,$$

where n is dimensions, $\ell = 100$ and $k = 200$;

$$a_m = \frac{1}{u - 2v} \int_{u}^{2v} f(x) \cos \frac{2m\pi}{2v - u} x dx;$$

to locate the optima of the origin function $f(x)$.

Since the auxiliary function $g(x)$ can enlarge the small attraction basins of the optima and flatten the large attraction basins, the $g(x)$ can guide an algorithm to search the optima with small attraction basins more efficient, and these optima are difficult to find in the original objective function by EAs. Furthermore, this strategy runs in parallel with first strategy and compensates the deficiency of the exclusion-based selection operators on the algorithm's risk of missing optima with many sharp attraction basins.

4 EFES: The Evolution Strategies with Exclusion-Based Selection Operators and a Fourier Series Auxiliary Function

In this section, we demonstrate how all the strategies developed in the previous sections could be embedded into the known evolutionary algorithms, to yield new versions of the algorithms. We will particularly take evolution strategies (ES) as an example. Consequently, a new type of ES the EFES will be developed. The incorporation of the developed strategies with other known evolutionary algorithms might be straightforward, and could be done similarly as that in the example presented below.

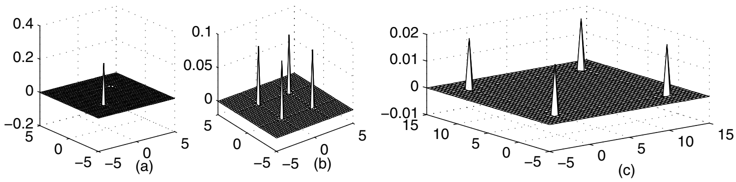

Fig. 2. A graphical illustration of the $g(x)$. (a) shows the complete Fourier system representation; (b) shows one element Fourier representation with parameters determined by $[u, v]$; (c) shows one element Fourier representation with parameters are determined by $[u, 2v]$.

The EFES Algorithm Is Given as Follows:

I. Initialization Step

I.1. Set $k = 0$, $\Omega^{(0)} = \Omega$, $n(0) = 1$ and $f^0_{best} = 10^8$; set the pre-determined number of cellular partition d, and stopping criteria ε_1 and ε_2, where ε_1 is the solution precision requirement and ε_2 is the space discretization precision tolerance.

I.2. Initialize all the ES parameters including:
$N-$ the population size;
$M-$ the maximum number of ES evolution.

II. Iteration Step (Epoch)

II.1.*ES Search in the auxiliary function $g(x)$:*
(a) randomly select N individuals from $\Omega^{(0)}$ to form an initial population;
(b) perform M steps of ES search, yielding the currently best minimum $g(x^*)$ in the region $\Omega^{(0)}$;
(c) if generation $= 1$, we put these points into population of ES for $f(x)$; if generation > 1, we will compare the fitness of $f(x)$ at these points with the current optimum of $f(x)$, and only put the points which values of $f(x)$ smaller than the current optimum of $f(x)$.

The two steps **II.2** and **II.3** below will use the cellular partition method, assuming $\Omega^{(k)}$ consists of $n(k)$ subregions, say, $\Omega^{(k)} = \Omega^{(k)}_1 \cup \Omega^{(k)}_2 \cup ... \cup \Omega^{(k)}_{n(k)}$.

For each subregion $\Omega^{(k)}_i$, do the following:

II.2.*ES Search in the objective function $f(x)$:*
(a) use the fixed points by step II.1 and some random selected points from $\Omega^{(k)}_i$ to form initial population;
(b) perform M steps of ES search, yielding the currently best minimum $f^{(k)}_i$ in the subregion $\Omega^{(k)}_i$;
(c) let $f^{(k)}_{best} := \min\{f^{(k)}_i, f^{(k-1)}_{best}\}$.

II.3. *Exclusion-based Selection:*
With the "best-so-far" fitness value $f^{(k)}_{best}$ guided, eliminate the "less prospective" individuals (cells) from $\Omega^{(k)}_i$ by employing appropriate exclusion operator(s) and a specific exclusion scheme. The remaining cells are denoted by $\Pi^{(k)}_i$.

II.4. *Space Shrinking:*

(a) generate a cell $\Omega_i^{(k+1)}$ such that $\Pi_i^{(k)} \subset \Omega_i^{(k+1)}$ and $\omega(\Omega_i^{(k+1)}) < \omega(\Omega_i^{(k)})$ whenever possible; in this case, set $\Omega_{i1}^{(k+1)} = \Omega_i^{(k+1)}$ and $\Omega_{i2}^{(k+1)} = \emptyset$;

(b) bisect $\Pi_i^{(k)}$ and construct two large cells $\Omega_{i1}^{(k+1)}$ and $\Omega_{i2}^{(k+1)}$ such that

$$\Pi_i^{(k)} \subset \Omega_{i1}^{(k+1)} \cup \Omega_{i2}^{(k+1)} \ and \ max\{\omega(\Omega_{i1}^{(k+1)}), \omega(\Omega_{i2}^{(k+1)})\} < \omega(\Pi_i^{(k)}).$$

III. Termination Test Step

If $\omega(\Omega^{(k)}) \leq \varepsilon_2$ and $|f^{(k)} - f^{(k-1)}| \leq \varepsilon_1$ hold for three consecutive iteration steps, then stop; Otherwise, go to step II with $k := k+1$, and

$$\Omega^{(k+1)} = \Omega_{i1}^{(k+1)} \cup \Omega_{i2}^{(k+1)} \cdots \cup \Omega_{n(k)}^{(k)}.$$

Detailed remarks on each step of the above algorithm are given in [6].

The iteration step (step II) is the core of the algorithm. Step II.1 performs the ES for M generations to search in the auxiliary function $g(x)$, the best individuals are potential global optima, which are difficult to find in $f(x)$. Adopting the comparison criterion could efficiently eliminate interference caused by other kinds of points (like discontinuous point). In order to ensure the safety of the algorithm, we do not apply the exclusion-based selection operators in the search process of $g(x)$.

Table 1. The test suite used in our experiments.

$f_1 = 4x_1^2 - 2.1x_1^4 + \frac{1}{3}x_1^6 + x_1x_2 - 4x_2^2 + 4x_2^4$, $n = 2,\ x \in [-5, 5]$;

$f_2 = x_2^2 + 2x_1^2 - 0.3cos(3\pi x_2) - 0.4cos(3\pi x_1) + 0.7,\ n = 2,\ x \in [-5.12, 5.12]$;

$f_3 = \sum_{i=1}^{n/4} (x_{4i-3} + 10x_{4i-2})^2 + 5(x_{4i-1} - x_{4i})^2 + (x_{4i-2} - 2x_{4i-1})^2 + 10(x_{4i-3} - x_{4i})^4, n = 40,\ x \in [-100, 100]$;

$f_4 = \sum_{i=1}^{n/4} 3[\exp(x_{4i-3} - x_{4i-2})^2 + 100(x_{4i-2} - x_{4i-1})^6 + [tan(x_{4i-1} - x_{4i})^4 + x_{4i-3}^8], n = 40,\ x \in [-4, 4]$;

$f_5 = \sum_{i=1}^{n/4} \{100[x_{4i-3}^2 - x_{4i-2}]^2 + (x_{4i-3} - 1)^2 + 90(x_{4i-1}^2 - x_{4i})^2 + 10.1[(x_{4i-2} - 1)^2 + (x_{4i} - 1)^2] + 19.8(x_{4i-2} - 1)(x_{4i} - 1)\},\ n = 40,\ x \in [-50, 50]$;

$f_6 = \frac{1}{4000} \sum_{i=1}^{n} x_i^2 - \prod_{i=1}^{n} \cos(\frac{x_i}{\sqrt{i}}) + 1,\ n = 40,\ x \in [-600, 600]$;

5 Simulations and Comparisons

We will experimentally evaluate the performance of the EFES, and compare it with the standard evolution strategies (SES). The EFES was implemented with $k = 10, N = 1000$ and $\varepsilon_1 = \varepsilon_2 = 10^{-8}$. The maximal number M of ES evolution was taken uniformly to be 500 on $f(x)$ and $g(x)$ respectively, called a epoch. All these parameters and schemes were kept invariant unless noted otherwise. For fairness of comparison, we also implemented the SES with the same parameter settings and the same initial population. The maximum number of SES evolution

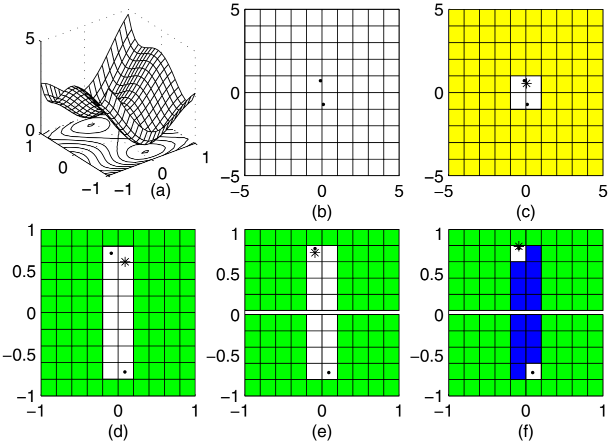

Fig. 3. The search space (the white cells) shrinking process when the EFES applied to f_1. (b) shows the locations of the global optima of f_1 (denoted by "•"); (c) and (d) show the cells excluded by the algorithm in epochs 1 and 2 (current best search point is denoted by "*". In epoch 3, the search space is forked into two subregions in (e) because the two global optima all lie in the husk layer of the search space. After forking, one more epoch yields the two global optima with precision 10^{-8}.

is 10^5. All experiments were run for ten times with random initial populations, and the averages of the ten runs were taken as the final result.

The test suite used in our experiments include those minimization problems listed in Table 1. The suite mainly contains some representative, complex, multimodal functions with many local optima and being highly nonseparable in features. Our experiments were divided into two groups with different purposes. We report the results of each group below.

Explanatory Experiments: This group of experiments aims to exhibit the evolution processes of the EFES in detail. To clearly demonstrate the running process of the exclusion-based selection operators, this simulation first studies the ES with exclusion-based selection operators and without the Fourier series auxiliary function to solve the problem f_1. Fig.3-(a) shows the function f_1 that has only two global minimizers. The evolution details (particularly, the search space shrinking details) of the EFES when applied to minimization of this function is presented in Figs.3 (b)-(f), which demonstrate clearly how the remaining cells are accumulated around the currently acceptable search point (denoted by "*") for the global minimum, and how local minima are successively excluded. This demonstrates the common features of the EFES. The experiments also demonstrate that the number of subregions, $n(k)$, contained in search space $\Omega^{(k)}$ in each step are uniformly bounded. These bounds are seen to be very small in each case, but vary with the problems under consideration. Particularly we observed in the experiments that the bounds $n(k)$ are generally related (actually, proportional) to the number of the global optima of the function to be optimized.

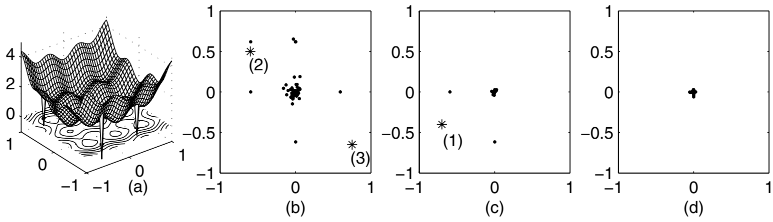

Fig. 4. A schematic illustration that the EFES to search on f_2. (a) shows random version of f_2. (b) and (c) show the results after a first and second epoches respectively. The points (1), (2) and (3) show the optima $(-1, 10^{-5})$, $(-0.5, 10^{-3})$ and $(-0.5, 10^{-3})$, respectively.

We designed the test function based on benchmark multimodal functions f_2 to demonstrate the feature of the auxiliary function $g(x)$. Three optima with small attraction basins are generated randomly in the feasible solution space of $f(x)$ and they have the following properties respectively: $(-0.5, 10^{-3})$, $(-0.5, 10^{-3})$ and $(-1, 10^{-5})$, representing the value of the optima and the width of its attraction basin in the bracket, respectively. The locations of these optima are decided in random.

Fig.4-(a) shows random version of f_2. Figs.4-(b) and (c) show the results of the EFES within the first and second epoches respectively, the points $*$ are obtained from the EFES in the search in $g(x)$. Fig.4-(b) demonstrates that the EFES can find the two optima $(-0.5, 10^{-3})$ in the first epoch(1000 generations), however the optimum $(-1, 10^{-5})$ is not represented in $g(x)$ at this time. In the second epoch, after bisecting the whole space between these two optima $(-0.5, 10^{-3})$, the optimum $(-1, 10^{-5})$ is represented by $g(x)$ and identified by the EFES(c.f. Fig.4-(c)). These results confirm that the EFES can find these three optima with small attraction basins in $g(x)$. Fig.4-(d) shows SES is converged to the optimum $f_1(0, 0) = 0$ after 10000 generations, which is the global optimum of the original version of function f_2. However, this point is not the global optimum of random version of f_2.

Comparisons: To assess the effectiveness and efficiency of the EFES, its performance is compared with the standard ES (SES). We define that $f_7 - f_{10}$ are the random versions of $f_3 - f_6$. Functions $f_7 - f_{10}$ have one optimum with a small attraction basin $(-1, 10^{-5})$, representing the value of the optimum and the width of its attraction basin in the bracket, respectively. The location of the optimum are decided in random. The comparisons are made in terms of the solution quality and computational efficiency and on the basis of applications of the algorithms to the test functions $f_3 - f_{10}$ in the test suite. As each algorithm has its associated overhead, a time measurement was taken as a fair indication of how effectively and efficiently each algorithm could solve the problems. The solution quality and computational efficiency are therefore respectively measured by the solution precision and the fitness attained by each algorithm within an equal period of fixed time. Unless mentioned otherwise, the time is measured in minutes as measured on the computer.

Table 2 and Fig.5 present the solution quality comparison results in terms of $f_{best}^{(t)}$ when the EFES and SES are applied to the test functions $f_3 - f_{10}$. We can observe that while the EFES consistently converges to the global optima for all test functions, SES is unable to find the global solutions for $f_3 - f_6$ and their random versions, $f_7 - f_{10}$. On the other hand, Table 2 and Fig.5 also show that the EFES can always locate the global optimum with higher solution precision. That is, the EFES outperforms SES in solution effectiveness.

The computational efficiency comparison results are shown in Fig.5. It is clear from these figures that the EFES significantly outperform the SES for all test functions. In addition, we could see from Figs.5 (e)-(h) that the efficiency increases because the auxiliary function $g(x)$ efficiently guide the EFES to find the global optimum with a small attraction basin. Even with such efficiency speed-up, the guaranteed monotonic convergence of the EFES is still clearly observed in all these experiments. All these comparisons show the superior performance of the EFES in efficacy and efficiency.

Table 2. The results of the EFES and SES when applied to test functions

Function	Epoches	The running time (minute)		The solution precision attained	
		SES	EFES	SES	EFES
f_3	5	17.2	17.2	10^{-4}	10^{-8}
f_4	5	19.0	19.0	10^{-3}	10^{-8}
f_5	5	21.6	21.6	10^{-3}	10^{-8}
f_6	5	16.8	16.8	10^{-3}	10^{-8}
f_7	3	17.2	7.3	10^{-4}	10^{-8}
f_8	2	19.0	5.8	10^{-3}	10^{-8}
f_9	3	21.6	15.8	10^{-3}	10^{-8}
f_{10}	3	16.8	9.1	10^{-3}	10^{-8}

6 Conclusion

In this paper, we have developed a new evolutionary algorithm— EFES, which incorporates two strategies, exclusion-based selection operators and the Fourier series auxiliary function into ES, to solve global optimization problems.

The EFES has been experimentally tested with a difficult test suite consisted of two groups of complex multimodal function optimization examples. The performance of the EFES is compared against the standard evolution strategies

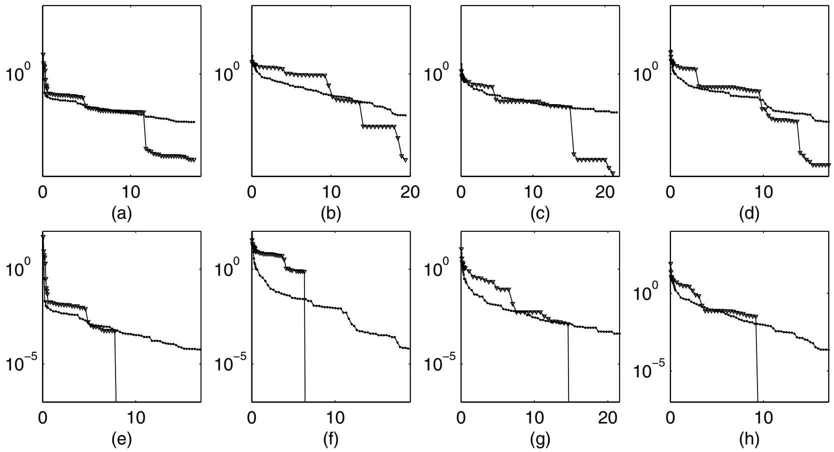

Fig. 5. The solution quality and computational efficiency comparisons of the EFES and SES when applied to problems $f_3(a)$, $f_4(b)$, $f_5(c)$, $f_6(d)$, $f_7(e)$, $f_8(f)$, $f_9(g)$, $f_{10}(h)$, where the abscissa is the time (minutes), and the ordinate is the solution precision $|f_{best}^{(t)} - f^*|$(absolute error)(Keys: v – EFES, . – SES).

(SES). All experiments have demonstrated that the EFES consistently and significantly outperforms the SES in efficiency and solution quality, particularly towards problems whose global optima are located in small attraction basins. Since the Fourier series auxiliary function could be used on discontinuous function optimization problems and prevent accidental deletion of the global optima with very narrow attraction basins by the exclusion-based selection operators, EFES has wider application, both for continuous and discontinuous problems.

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