Computation and Computability

S. Barry Cooper
School of Mathematics, University of Leeds
Leeds, LS2 9JT, UK
www.maths.leeds.ac.uk/˜pmt6sbc
s.b.cooper@leeds.ac.uk

The last century saw dramatic challenges to the Laplacian predictability which had underpinned scientific research for around 300 years. Basic to this was Alan Turing's [10] 1936 discovery (along with Alonzo Church) of the existence of unsolvable problems.

To a classical computability theorist, one of the more striking things about the the 1936 discoveries and their repercussions was that the emergence of incomputability, soon after that of quantum unpredictability, with its iconoclastic import for old certainties, was hardly acknowledged outside the hermetic confines of the academic world. At the same time, the universal Turing machine, a mere by-product of that unexpected glimpse beyond Laplacian predictability, became the basis of an informational revolution built on filling out the computable.

Of course, 1936 not only saw a new clarity about what 'incomputability' really is, but also the emergence of a conceptual framework which actually took people *away* from the real world and the uncertainties facing working scientists. Sometimes, being precise about notions has negative as well as positive consequences. It may make them technically more useful, while disrupting their more general usefulness. There were also mathematical developments which suggested incomputability might be just as containable as sub-atomic unpredictability had become within quantum theory. John Myhill [6] showed that all the unsolvable problems discovered in the 1930s were essentially *the same*. They were just notational variants of what he called a *creative* set. One could artificially manufacture other incomputable objects, but there was no evidence of their existence outside of academic papers. Anyway, there seemed little point in worrying about the recursion theorists' negative news, when there was so much computability to explore. The richness of the *computable* universe was increasingly revealed, and provided more than enough work for a whole army of researchers.

Things are changing. What we seem to see now is the beginning of a new confluence of theoretical and pragmatic approaches to questions concerning computation and computability.

Earlier mathematical pointers in this direction included the negative solution to Hilbert's Tenth Problem (due to Martin Davis, Yuri Matiyasevich, Hilary Putnam, and Julia Robinson), which demonstrated that all the incomputable sets which could be artificially enumerated by clever logical techniques were not artificial at all — they arose *naturally* as solution sets of the familiar diophantine equations. Everyday mathematics leads us unavoidably to incomputable objects.

For the working scientist, naturalness is captured within a rather different kind of mathematics, full of real numbers and differential equations. Marian Pour-El was one of the first to look for appropriate notions and evidence of incomputability arising from this context. One of the most widely-known and enduring discoveries, due to Pour-El and Richards [7], was a differential equation with computable boundary conditions leading to incomputable solutions. More recently, high-profile names (such as Roger Penrose, Steve Smale) have been associated with investigations of the computability of well-known mathematical objects with known connections with complexity in nature, such as the Mandelbrot and Julia sets.

The closer one gets to physical reality, the more potentially persuasive, while at the same time more speculative and ultimately elusive, the examples become. Apart from the workings of the human brain, the most scrutinised and puzzling part of the physical world is the sub-atomic level. The predictive incompleteness of quantum theory refuses to go away, giving rise to different 'interpretations' which leave us a long way from characterising the algorithmic content of the events it seeks to describe. The quantum process which seems to escape the predictive net most radically, the most promising avatar of incomputability, is by-passed by current quantum computational models — despite recent claims (see Tien Kieu [4]). This is Andrew Hodges' comment on the situation (taken from his article *What would Alan Turing have done after 1954?*, in Teuscher [9]):

> *"Von Neumann's axioms distinguished the* **U** *(unitary evolution) and* **R** *(reduction) rules of quantum mechanics. Now, quantum computing so far (in the work of Feynman, Deutsch, Shor, etc) is based on the* **U** *process and so computable. It has not made serious use of the* **R** *process: the unpredictable element that comes in with reduction, measurement, or collapse of the wave function."*

Observable signs of incomputability in nature are not so obvious at the classical level - we are more embroiled and cannot so easily objectify what we are part of. As early as 1970, Georg Kreisel was not deterred, and in a footnote to [5] (p.143) went so far as to propose the possibility of a collision problem related to the 3-body problem which might give "an analog computation of a non-recursive function (by repeating collision experiments sufficiently often)".

This conjecture has come to seem less outrageous as people from various backgrounds have come up with even more basic proposals. One of these comes out of recent progress on the Painlevé Problem, from 1897, asking whether noncollision singularities exist for the N-body problem for $N \geq 4$. For $N \geq 5$, Jeff Xia in 1988 showed the answer is "Yes" (see Saari and Xia [8]). And actual, or even uncompleted, infinities in nature are what open the door to incomputability. There are a number of other recent examples.

Recently attention has turned to Turing's universe of computably related reals (based on his oracle machines intoduced in [11]) as providing a model for scientific descriptions of a computationally complex real universe (see [1], [2], [3], etc.) This is based on a growing appreciation of how algorithmic content brings with it implicit infinities, and a science — increasingly coming to terms with chaotic and non-local phenomena — *necessarily* framed in terms of reals rather than within some discrete or even finite mathematical model. There is, of course, a huge amount of research activity, much of it ad hoc in nature, concerned with the computational significance of evolutionary and emergent form, and emergence in more specific contexts. There is a role here for the unifying and clarifying role of basic mathematical structures.

What is currently exciting to many of us is that the sorts of questions which preoccupied Turing, and the very basic extra-disciplinary thinking which he brought to the area, are being revisited and renewed by researchers from quite diverse backgrounds. What we are seeing is an emergent coming together of logicians, computer scientists, theoretical physicists, people from the life sciences, and the humanities and beyond, around an intellectually coherent set of computability-related problems. The recurring and closely linked themes here are the relationship between the local and the global, the nature of the physical world, and within that the human mind, as a computing instrument, and our expanding concept of what may be practically computable.

The specific form in which these themes become manifest are quite varied. For some there is a direct interest in incomputability in Nature, such as that coming out of the n-body problem or quantum phenomena. For others it is through addressing problems computing with reals and with scientific computing. The possibility of computations 'beyond the Turing barrier' leads to the study of analog computers, while theoretical models of hypercomputation figure in heated cross-disciplinary controversies. Perhaps most importantly, there is intensive research going on into a number of practical models of natural computing, which present new paradigms of computing whose exact content is as yet not fully understood. In many scientific areas the emergence of form is deeply puzzling, and there is a need for mathematical models.

What is taking shape seems to be the 1936 paradigm shift renewed. In the 1920s and 1930s we saw a fracturing of the comfortable picture of how science could bring predictability to a complex universe, unaccompanied by any overall concept of the underlying mathematical structures. We now see a coming together (at times faltering, at times confused) of science and mathematics to replace the Laplacian model of science with one whose complexities match those of the real world. At the root of this is both the technical legacy of Turing, and the kind of unified approach to scientific problems that was so characteristic of Turing's own thinking.

The potentially unifying role of the classical computability theorist here is, as yet, peripheral. But the separation since Turing between computability theory and its real-world counterpart, is being slowly repaired.

REFERENCES

[1] Cooper S. B. (1999) Clockwork or Turing U/universe? – remarks on causal determinism and computability. In *Models and Computability* (S. B. Cooper and J. K. Truss, eds.), London Mathematical Society Lecture Note Series 259, Cambridge University Press, Cambridge, pp. 63–116.

[2] Cooper S. B. and Odifreddi P. (2003) Incomputability in Nature. In *Computability and Models: Perspectives East and West* (Cooper S. B. and Goncharov S. S., eds.), Kluwer Academic/ Plenum Publishers, New York, Boston, Dordrecht, London, Moscow, pp. 137–160.

[3] Copeland J. (1998) Turing's O-machines, Penrose, Searle, and the brain. *Analysis*, **58**, 128–38.

[4] Kieu T. (2003) Quantum algorithm for the Hilbert's Tenth Problem. *Int. J. Theoretical Physics* **42**, 1461–1478.

[5] Kreisel G. (1970) Church's Thesis: a kind of reducibility axiom for constructive mathematics. In *Intuitionism and Proof Theory: Proceedings of the Summer Conference at Buffalo N.Y. 1968* (Kino A., Myhill J., and Vesley R. E., eds.), North-Holland, Amsterdam, London, pp. 121–150.

[6] Myhill J. (1955) Creative sets. *Z. Math. Logik Grundlag. Math.* **1**, 97–108.

[7] Pour-El M. B. and Richards J. I. (1983) Noncomputability in analysis and physics. *Adv. Math.* **48**, 44–74.

[8] Saari D. G. and Xia Z. (Jeff) (1995) Off to infinity in finite time. *Notices of the Amer. Math. Soc.* **42**, 538–546.

[9] Teuscher C. (ed.) (2004) *Alan Turing: Life and legacy of a great thinker*. Springer-Verlag, Berlin, Heidelberg.

[10] Turing A. M. (1936) On computable numbers, with an application to the Entscheidungsproblem. *Proc. London Math. Soc.* (2) **42** (1936–7), pp. 230–265. Reprinted in A. M. Turing, *Collected Works: Mathematical Logic*, pp. 18–53.

[11] Turing A. M. (1939) Systems of logic based on ordinals. *Proc. London Math. Soc.* (2) **45**, pp. 161–228. Reprinted in A. M. Turing, *Collected Works: Mathematical Logic*, pp. 81–148.