Schelling's Bounded Neighbourhood Model: A systematic investigation

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Abstract

"Essentially, all models are wrong, but some are useful." George E. P. Box [20]

This thesis explores the role of modelling and computational simulation, in relation to social systems, with specific focus on Schelling's seminal models of segregation. It discusses the role of computational modelling and some techniques that can be used in the Social sciences. Computational modelling is potentially powerful; however, the complexity of the domains being modelled means caution is needed. Simulation of social interaction has consistently created debate in the Social sciences. From the differential equations of predator-prey models, to Agent Based Models of Schelling's segregation, most models are dismissed as either too simplistic or unrealistic. In an attempt to counter these criticisms, more complex models have been developed. However, by increasing the complexity of the model, the underlying dynamics can be lost. Schelling's models of segregation are a classic example, with much of the work building on his simple segregation model. The complexity of the models being developed are such that, real world implications are being inferred from the results. The Complex Systems Modelling and Simulation (CoSMoS) process has a proven track record in developing simulations of complex models. In a novel application, the CoSMoS process is applied to Schelling's Bounded Neighbourhood Model. The process formalises Schelling's Bounded Neighbourhood Model and, from the formalisation, develops a simulation. The simulation is validated against the results from Schelling's model and then used to question the model. The questioning of the model is an attempt to examine the underlying dynamics of the segregation model. In this respect, two measures, static, regarding the final population mix (M), and dynamic, counting the number of iterations (I), are used in the analysis of the results. In the initial experiment, the effect of ordered movement was tested by changing the movement, from ordered to random. A second experiment examined agents' perfect knowledge of the system. By introducing a sample size (S), the agents' knowledge of the system is reduced. The third experiment introduced a friction parameter (F), to examine the effect of ease of movement into and out of the neighbourhood. In the final experiment, Schelling's model is recast as a network model. Although the recasting of the model is slightly unorthodox, it opens the model up to network analysis. This analysis allows the easy definition of a 'social network' that is overlaid on Schelling's 'neighbourhood network'. Two different networks are applied, Random and Small World, with the size of the network controlled by m_s. The results of the experiments showed, that Schelling's model is remarkably robust. Whilst the adjustments to the model all contributed to changes in the output, the only significant difference occurred when the social network was added.

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Authour's declaration

The work presented here is all my own, except where otherwise referenced.

Chapter 1

Introduction

Advances in computing, both in terms of speed and programming, have made an impact in many areas. The linking of computing with natural sciences has produced fields such as quantum computing, genetic algorithms and chemical computing. Economics is attempting to exploit this computational power, with current research based on developing Artificial Intelligence that can take advantage of the market and make profits. Computation has also spread into the social sciences: Human Computer Interaction (HCI) considers the effects of computing on individuals, usability and statistical methods; History, where multi-agent systems have been used to model counter-factual events and Sociology where multi-agent systems are used to create artificial societies [2]. Many of these fields share common ground with their use of modelling and simulation to investigate complex systems and complexity.

According to Aristotle

> "the whole is more than the sum of the parts." [9]

This is considered to be the first known attempt to describe the outcome of a complex system [61]. More than 2000 years later, there is still no satisfactory scientific definition, but a great deal of debate. For now, a complex system is defined, broadly speaking, as *a system containing components that interact, producing global behaviour that are something other than the aggregate of those interactions.* Complexity is taken to be the study of complex systems and their emergent behaviour. Descriptions of complex systems range from Aristotle's sentence, to the mathematical entropy of Shannon [81]. In a review of the prevalence of complexity, Eve *et al.* [33] offer a list, summarized in Table 1.1. This list, which omits economics and politics, illustrates the abundance of complex systems in the natural and social sciences. It should be noted that while this trans-disciplinary approach links previously unrelated research areas, it causes controversy over fundamental issues such as definitions (ontology) and

Subject area	**Examples of complexity**
Mathematics	an extension of limit theory and complex topology [22].
Classical Physics	problems of turbulence [51].
Thermodynamics	issues in the study of entropy [55].
Chemistry	refinements of catalysts + phase boundaries [44].
Biology	description of ecological feedback [25].
Sociology	bio-mass use in a region [77].

Table 1.1: A table of examples (with additional entries for Biology and Sociology) of complexity by subject area from [33].

methodology. Certainly there is no universally agreed definition of what constitutes a complex system. These debates over complex systems, whilst important, can distract from the task at hand. To avoid this issue, here a complex system is defined as: a system containing components that interact, producing global behaviour that are something other than the aggregate of those interactions.

Within the area of modelling and simulation, the advances in computational power have led to numerous applications. This can be seen in the work of Schelling in his segregation models [74] and Epstein's Sugarscape [31], both of which have been used in much of the discourse on sociology and simulation. Both these experiments were motivated by real world social systems. These social simulations are an attempt to reproduce the complexity of real world human systems from models of core processes that underlie global behaviour. However, the view of social systems as complex systems is far from accepted within social science. Yet, one can argue the case that, social systems have many characteristics in common with complex systems. Indeed, a growing number of authours, especially in relation to cities and urban regions, have made this point (for example [21] [43], [12] and [68]). These authours (and others) have begun to use complexity to describe and model social systems. Some of these techniques are introduced in Chapter 2, with examples of their application to social systems. The most common architectures used is Agent Based Modelling (ABM). One of the most well know of these models is Schelling's segregation model, which is discussed in Chapter 3. The model has been used to describe cities in both America and Israel, and shows that slight intolerance leads to high levels of segregation. These findings have led to heated debates about the validity of the model, and ideas of validity and verification are the focus of Chapter 4. Following Epstein's argument [30], that a general ABM framework will emerge from individual frameworks, the Complex System Modelling and Simulation (CoSMoS) process is applied to Schelling's model. The CoSMoS process (discussed in Section 4.3) has successfully developed valid simulations of complex systems. In a novel application, the process is applied to Schelling's Bounded Neighbourhood Model. The development of a valid simulation of Schelling's Bounded Neighbourhood Model is described in Chapter 5. Once validated, the simulation is used to test Schelling's ordered movement rule, the work is presented in Chapter 6. Following the CoSMoS process, the results from the experiment lead to more questions about the model. From this three assumptions are examined: Chapter 7 explores the effects of agents' complete knowledge of the system; Chapter 8 restricts the movement of the agents; whilst Chapter 9 recasts the model as a network and introduces a social network. A summary of the results and contributions of the thesis, as well as some conclusions and further work, are presented in Chapter 10.

Chapter 2

Modelling Social Systems

Although there is disagreement about the structural aspects of social systems, a minimal requirement is the interaction of two organisms [100]. There is agreement that social systems have many characteristics in keeping with complex systems [73]. From this stems the belief that the tools and techniques of complexity can be applied to social systems. Although the field of complexity could be seen to have strong links to sociology [21], the ideas are still far from accepted by many in the social sciences. Authours such as Sawyer [73], Byrne [21], Fossett [37] and Walby [95] are suggesting that the tools of complexity offer a great deal to social science. Some of the tools are now presented, with brief examples of their application to social systems.

2.1 Differential Equations

Differential Equations are at the heart of mathematical physics and are a significant motivation for mathematical analysis [82]. Differential equations involve the relationship between infinitesimal changes of independent variables (such as time) and the dependent variables such as temperature. In the simplest case, given a 'smooth' function $y = f(x)$, its derivative $f'(x) = dy/dx$ can be interpreted as the rate of change of y with respect to x. Many of the general laws of nature, such as physics, chemistry, biology and astronomy, can be expressed in this language. In the physical sciences the equations are developed from a deep and precise understanding of the phenomena being studied. It is effective where the constituent entities are homogeneous, deterministic and smooth (at least statistically).

The Italian mathematician Volterra, known for his work on integral equations, used differential equations to model biological systems [92]. This mathematical biology led Alfred Lotka to theorise that natural selection was a struggle for available energy. Those organisms that capture and use energy most efficiently are the ones that prosper. (It is interesting that he suggested the switch in society from solar to non-renewable energy would pose a fundamental challenge [5]). Applying these ideas of efficiency to animal populations led to the development of the Lotka-Volterra predator prey model. The equations model the growth and decline of two populations, one predator (often referred to as foxes) and one prey (often rabbits). Let $R(t)$ and $F(t)$ represent the number of rabbits and foxes respectively that are alive at time t, then the Lotka-Volterra model [86] is:

$$\begin{aligned}\frac{dR}{dt} &= aR - bRF,\\ \frac{dF}{dt} &= eRF - cF,\end{aligned}$$

where a is the natural growth rate of rabbits in the absence of predation (foxes), b is the death rate per encounter of rabbits due to predation (foxes), c is the natural death rate of foxes in the absence of food (rabbits), and e is the efficiency of turning predated rabbits into fox's offspring. The equations are part of a family of models known as logistic growth equations, other types include the parasitic model and the competition model. The parameters selected are considered to be the most important aspects of the system being analysed. Certainly birth and death rates are the most important factor in population size and these are clearly affected by the presence (or absence) of food. In the model the number of encounters between the two groups is found to be proportional to the number of rabbits times the number of foxes. This would seem to imply there is no spatial segregation of the populations as they appear to consistently encounter each other. Despite their relative simplicity these equations have been successful at modelling pairs of populations such as lynxs and hares in the Hudson Bay [54] and, according to Kauffmann [50, p210], arctic foxes and hares as well as commercial fishing in the Adriatic.

In an influential paper, Hosler *et al.* [48] used differential equations to model the collapse of the Mayan civilization between 750 - 900AD. Basing their work on the anthropological theory put forward by Willey and Shimkin [101], they developed a model based on a deviation-amplifying feedback loop, between the increasing ritual and public building. This enhanced the prestige of the elite, led to the consequent decline in food production, which was considered demeaning [48]. These and other elements, of the social and economic systems that made up the society, were quantified. From these, specified recurrence equations[1] were

[1] "A recurrence equation (or difference equation) is the discrete analog of a differential equation. A difference equation involves an integer function $f(n)$ in a form like $f(n) - f(n-1) = g(n)$ where g is some integer function. This equation is the discrete analog of the first-order ordinary differential equation $f'(x) = g(x)$." [59].

chosen, to determine their interacting behaviour over time. The model produced showed a collapse, and further, that different behaviours from the ruling elite, could have postponed the collapse (although not prevent it). However, it is important to note that, in his critique, Gilbert [43, p42] points out this model completely ignores some, seemingly important, factors such as interactions with other tribes, warfare and spatial factors. Indeed later studies showed the Mayan 'collapse' was less suddenm and pointed towards a more gradual shift of dominance from one part of the region to another. This result exemplifies the danger of producing models with inaccurate data, and without careful consideration of the techniques being used. It certainly shows that, when modelling complex systems, even the 'correct' result can sometimes be 'wrong'. Often differential equations, or some variation such as difference equations, are more than adequate for simulating dynamical systems. But, if the system being measured is non-linear, even the smallest of variations in the initial settings can have an exponential effect on the outcome (sensitivity to initial data or 'Butterfly Effect [86]). This problem has become well known and can be traced back to Lorenz's attempts to model the weather using a computer simulation based on the following simple system of three ordinary coupled differential equations that describe the relevant atmospheric physics [86].

$$\begin{aligned}\frac{dx}{dt} &= \sigma(y-x)\\ \frac{dy}{dt} &= x(\phi-z)-y\\ \frac{dz}{dt} &= xy-\beta z,\end{aligned}$$

where $\sigma, \phi, \beta > 0$ are physical constants.[2] After Lorenz repeated his calculations from an earlier data point, he noticed significant deviations developed in time even though calculation were restarted from the exact same point in the plot. This was the origin of the familiar Butterfly Effect, as apparently insignificant rounding errors correspond to the effect that the motion of a butterfly's wings could have on the weather.

Differential equations are often used to investigate systems in the natural sciences. However they are unable to capture individual behaviours of members of a population being modelled. Additionally they are only able to incorporate spatial aspects of the system by

[2]for the interested reader σ is the Prandtl number which "...is a dimensionless number approximating the ratio of momentum diffusivity (kinematic viscosity) and thermal diffusivity. It is named after the German physicist Ludwig Prandtl" [97]. ϕ is the Rayleigh number which "...is a dimensionless number associated with buoyancy driven flow (also known as free convection or natural convection). When the Rayleigh number is below the critical value for that fluid, heat transfer is primarily in the form of conduction; when it exceeds the critical value, heat transfer is primarily in the form of convection" [98].

introducing further variables. This leads to more intractable partial differential equations. Moreover, while models of complex systems using differential equations can be accurate over small timescales, their nature means that any real long term predictions suffer from exponential error rates. Parameters selected for modelling in the natural sciences are usually very simplistic and are, perhaps, chosen more for mathematical simplicity. For example in the predator-prey model, the parameter that relates the food obtained to the number of offspring is hardly a realistic representation. The offspring are the result of far more complex interactions than just the amount of food available. Introducing more parameters to make the simulation more realistic would obviously make the equations more complicated. Moreover some natural aspects such as hunting or escape strategies cannot be incorporated in a simple way into numerical equations. There is no way, at present, of introducing modifications to the environment; the results of external changes can only be seen in terms of population numbers. Drogoul and Ferber [27] point this out, arguing that, with numerical modelling, the hunting/feeding process can only be represented by the volume of available prey, and the probability of the predator finding the prey. Rather than describing the behaviour of the predators/prey, it can only quantify the relation between the numbers involved. Finally, numerical simulations are unable to cope with qualitative data, such as the relation between a stimulus and the behaviour of an individual, which are well beyond the scope of analytical equations and numerical simulations [27]. Perhaps most important, there seems to be no way of capturing the evolution of 'emergent' behaviour or structure. Whilst modelling complex systems using differential equations offers a reasonable representation at a population level, it is clear that there are limitations that need to be taken into consideration. Most importantly, dealing with qualitative data is proving intractable in models for the natural and social sciences. Nevertheless, despite this, some simpler complex systems can be represented by a relatively small number of key variables.

2.2 Cellular Automata

Cellular Automata (CAs) were developed by von Neumann and Ulam [94] in the 1940s. In collaboration with another mathematician Ulam, von Neumann had been searching for logical solutions to the complexity of natural growth [99]. Von Neumann demonstrated that, on a two dimensional lattice grid, particular patterns of cells, with 29 states and specific state transitions, could self-replicate. It was discoverd that simple rules, governing state transitions, can create a variety of unexpected behaviours. Wolfram [104] used a simple form of CA, what he called an 'elementary cellular automata'. Elementary CAs have two possible values for each cell (0 or 1), and rules that depend only on the values of the cells directly

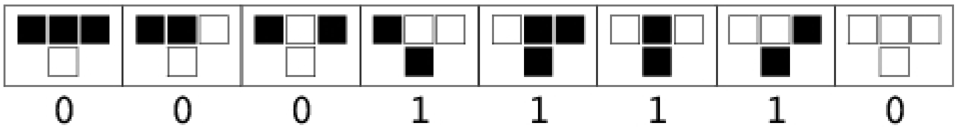


Figure 2.1: Wolfram's binary representation of 30

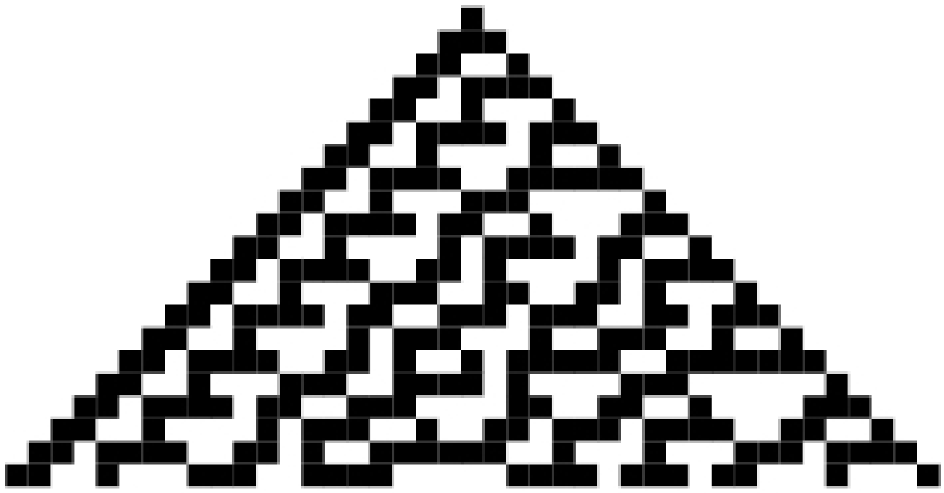


Figure 2.2: The first 15 generations of rule 30

adjacent. The evolution of an elementary cellular automaton can completely be described by a table specifying the state a given cell will have in the next generation based on the value of the cell to its left, the value the cell itself, and the value of the cell to its right. Since there are $2 \times 2 \times 2 = 2^3 = 8$ possible binary states for the three cells neighbouring a given cell, there are a total of $2^8 = 256$ elementary cellular automata, each of which can be indexed with an 8-bit binary number. An example of 30 (called rule 30) can be seen in Figure 2.1.

Each generation of cells calculates its state in the next generation, which is then displayed on the row below. Some of Wolfram's rules reached an equilibrium in a few generations and 88 produced results that were mirror images. However, the most interesting was rule 30 (Figure 2.1) which produced a chaotic result (Figure 2.2). This 'chaotic' behaviour was used by Wolfram to generate random numbers for large integers in Mathematica [105].

The interesting behaviours of CAs have led to numerous different experiments with different parameters. Adjustments to the shape of the cells has led to triangular and hexagonal CAs, whilst an increase in the size of possible systems allowed Rendell [70] to build a universal Turing machine in the 'Game of Life'.

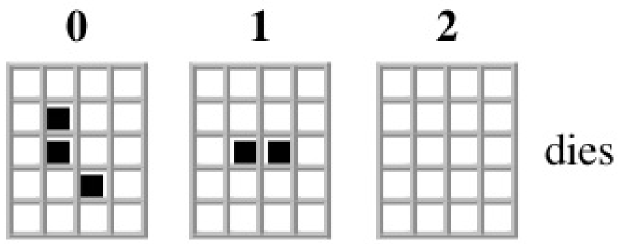


Figure 2.3: An example run of the Game of Life.

The Game of Life [40], developed by John Conway in the 1970s, is often used as an example of a complex system; since many argue that it has emergent behaviours. Conway was searching for a simple ruleset that could produce interesting behaviour, using von Neumann's [94] cellular automata architecture. Conway explored a simplified version in the same two dimensional environment. Rather than the 29 states of von Neumann, Conway's cells had just two states: active or inactive. The state of a cell is based on the state of its immediate neighbourhood, defined as the eight surrounding squares. A simple ruleset governs the transition from the current state to the next state.

- **Active cells**
  - with two or three neighbouring active cells survives for the next generation.
  - with four or more active neighbours dies (becomes inactive) from overpopulation.
  - with less than two neighbours dies from isolation.
- **Inactive cells**
  - if adjacent to exactly three active cells becomes a birth cell and will be active at the next generation.

Conway was searching for a simple initial configuration that would result in complex behaviour. Whilst most configurations quickly moved to equilibrium in very few generations (for example Figure 2.3), one configuration produced some startling results. From the initial r-pentomino configuration (Figure 2.4) the system completes over 1000 generations before reaching a stable state. Small groups of five active cells undergoing simultaneous reflection and translation give the appearance of moving structures (Figure 2.5). These structures are called gliders from this reflection and translation known as glide symmetry. Other cells get caught in cyclic (and hence essentially finite) configurations which appear to blink on and

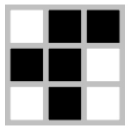


Figure 2.4: The r-pentomino.

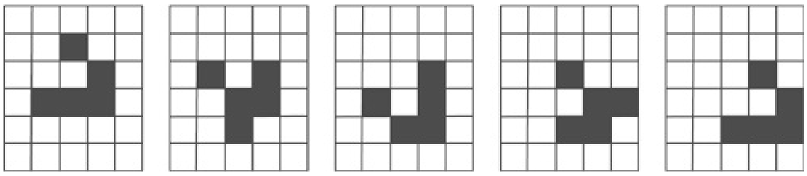


Figure 2.5: The reflection/translation action of the 'glider' gives the appearance of a structure moving across the environment.

off. Solid structures are formed by groups of cells. From the ruleset one might anticipate the blocks and even the 'blinker' structure. However the 'gliders' seem to be impossible to predict from the rules without explicitly simulating them. Their appearance has been called emergent by a number of authours, including Langton [52] and Bedau [13], and correspond to our description of **emergent properties** of the system. Of course a perception of a system level is needed to detect them (or their motion) since at the cellular level all one can detect is an active/inactive cell.

The Game of Life has been the focus of a number of studies, see for example one based on statistical analysis by Schulman [80]. Bak et al [11] use "statistical mechanics upon the self-organised systems" in an attempt to describe behaviours of groups of cells (such as gliders), rather than individual cells. The description can be seen as using the global emergent properties to measure the system. Bak notes that a trait of self-organising systems is their scale free structures. Scale free properties follow a power law distribution that applies at any scale. Such that, given a function $f(x) = ax^k$, scaling that by a factor of c causes only a proportionate scaling of the function, thus:

$$f(cx) = a(cx)^k = c^k f(x) \propto f(x).$$

Bak uses a 150×150 lattice grid with open boundary conditions and initialises the system with an (unspecified) random distribution of live sites. Bak then defines a measure s of the total activity of the system and allows the system to evolve until it reaches a 'rest state'. The rest state of the system is defined as "local still life and simple cyclic life" [11] and ignores long period cyclic states since Bak concedes his "method of generation cyclic structures of long period are extremely rare and essentially never encountered" (Bak does not define what the length of a long period is). Once the system is at rest, Bak perturbs it (without saying what form the perturbation takes) and measures the activity that arises. This activity s is the cumulative number of 'births' and 'deaths' from the initial rest state to the next rest state. By averaging 40,000 such perturbations Bak finds that, not only does the distribution of the activity of the system $D(s)$ follow a power law $D(s) \propto s^{-\tau}$ for $\tau \approx 1.4$, but so does the distribution of the duration of the perturbations, $D(T) \propto T^{-b}$ for $b \approx 1.6$ [11]. By ignoring the low level interactions and instead focusing on the emergent behaviour of the system, Bak is able to neatly side step having to track a possibly exponentially growing number of micro-states. Since each cell can have two possible states n cells need to consider 2^n possible states, this means the possibles number of states for Bak's experimental setup is 2^{22500}. The problem is akin to the gas laws, whereby, whilst it is possible to work out the temperature, one could never know the individual velocities of the molecules. So Bak *et al.* resort to measuring the emergent properties rather than calculating individual cell positions and states. The complexity the Game of Life produces from such simple rules offers the *possibility* that, some of the complexities of social interaction could be reduced to a few simple rules.

2.3 Game Theory

Game theory is a well known field of mathematics and was introduced by John von Neumann and Oskar Morgenstern in their influential book "Theory of Games and Economic behaviour" [93]. Game theory is the formal study of interactions between 'goal-oriented' agents and the strategic scenarios that arise in these settings. Further details can be found in von Neumann's book [93]. However, briefly, players interact through games of chance. Their strategies are the executions of choices, based on a payoff and probability. Players work towards maximizing their payoffs by assuming all players will be doing the same and thus attempting to develop a 'best response' to a rational choice.

A famous example of modelling using game theory is the prisoners' dilemma. In this problem two players (A and B) face a choice of betraying or co-operating. If both choose to co-operate their payoff is 2, if they both betray the payoff is 1. The dilemma comes from the fact that if one player betrays whilst the other co-operates the betraying player gets a payoff of 3 whilst the co-operator gets nothing. Consider player A: if B co-operates, A does better by betraying (3) than cooperating (2); if B betrays, A does better by betraying than co-operating (0). Therefore, whatever happens, the best response for both players is to betray, even though they would both be better off if they both co-operated.

Zhang [109] uses Game Theory to model Schelling's Bounded Neighbourhood Model. His model is a lattice graph with periodic boundary conditions (i.e a torus), where each vertex is a residential location. Any location i has H neighbouring vertices, where H is a fixed integer. So that $H = 4$ would be a von Neumann neighbourhood and $H = 8$ is a Moore neighbourhood. P_i is the price of location i, W_i the number of White neighbours and B_i the number of Black neighbours. Zhang assumes all the agents earn an identical income Y, since he claims that "even if each person i earns a different Y_i, all results remain the same" [108]. The price P_i is determined by a simplified market mechanism whereby prices respond to demand, more demand implies higher prices, so that $P_i = B_i + W_i$. From this Zhang is able to define a utility of a White agent as $U_{wi} = Y + \pi W_i - P_i = Y + (\pi - 1)W_i - B_i$ where $\pi > 0$. A positive value of π implies the more White neighbours a White agent has the happier the agent is. Letting $\theta = \pi - 1 > -1$ gives

$$U_{wi} = Y + \theta W_i - B_i.$$

Zhang now assumes Blacks care only about price at location j giving them a utility,

$$\begin{aligned} U_{bj} &= Y - P_j \\ &= Y - W_j - B_j. \end{aligned}$$

From this utility Zhang makes a proposition:

> If Blacks are colour-neutral and Whites have a slight preference for like-colour neighbours, then, in the long run: (i) residential segregation is observed most of the time; (ii) the rate of vacancy is higher in Black neighbourhoods than in White neighbourhoods; and (iii) Whites pay more than Blacks do for equivalent housing.

Zhang's model points to this proposition, since Whites would be unwilling to move to a black neighbourhood, whereas blacks would have no problem moving. Although Zhang introduces a bounded rationality, in that agents will sometimes make utility decreasing moves, the effect is negated over the time periods he uses. The erroneous moves are eventually outweighed by rational movement. Zhang's P_i equation assumes location i is occupied because Zhang believes it odd that i commands a different price before and after an agent moves in. No evidence to support this assumption is given (and if one considers a property that has been vacant for a long period of time, the price could increase if it became occupied). It is possible that Zhang would consider the change in price to be led by the market as a whole rather than an individual in the market. Additionally the model uses rental housing market data to define the pricing mechanism. Zhang fails to experiment with the preference parameters to see the

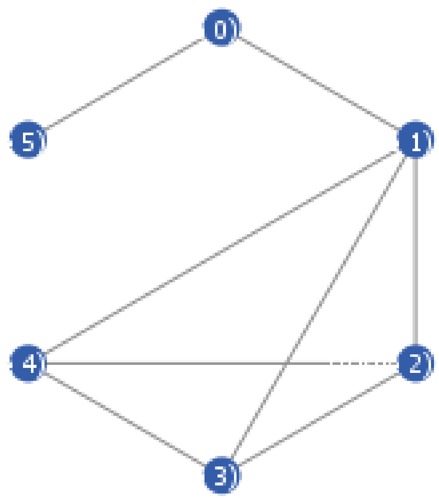


Figure 2.6: The network with $n = 6$ and $m = 8$.

effect they have, mainly citing the fact the model is using 'some' survey data from Farley *et al.* [35] and [34]. Whilst Farley's work would seem to suggest Zhang's conclusions about preference is correct, Zhang could have explored the results of different preference levels, to see if Farley's survey work still held for his model.

Whilst game theory is a useful tool in modelling social interactions, it creates an environment specific to the problem domain. There is no concept of any other interactions or the effects they might have on the results. Certainly in Zhang's experiment there is no evolution of agents, the population is static and the evolution of the agents themselves is an important part of any model of society. Indeed, as van Baal [90] points out, a game theoretic model takes a hugely simplistic approach, even unrealistic, as often agents involved play a huge number of 'games' to evolve a strategy without really implementing any individual evolution.

2.4 Networks

From the biological to the social, network descriptions have been used to unlock insights into the systems they are describing.The tools used in the analysis of such networks have been developed from graph theory (the mathematical field that deals with networks). Consider the network $G(V, E)$, where V is a set of nodes (verticies in graph theory, hence the V) and E a set of unordered pairs of members of V here called links (edges in graph theory). The cardinality of V is called the order of G and (following [63]) is denoted n, whilst the cardinality of E is called the size of G, denoted here by m. A simple network is shown in Figure 2.6. Whilst Figure 2.6 is a nice visual representation of a network, it is difficult to analyse mathematically. Another representation is the *adjacency matrix* $\mathbf{A}$. For example, creating a matrix based on

$$A_{ij} = \begin{cases} 1 & \text{if there is a link between nodes } i \text{ and } j \\ 0 & \text{otherwise;} \end{cases} \tag{2.1}$$

means, the network shown in Figure 2.6 can also be represented in the following manner:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.2}$$

The matrix retains all the information from the graph and submits easily to analysis. For example just by looking at the matrix two things are apparent. Firstly, $\mathbf{A}$ is zero along the diagonal; this means there are no edges from a node to itself. Secondly, $\mathbf{A}$ is symmetrical, meaning the network is undirected (*i.e.*, if there is a link between i and j, then there is a link between j and i). In addition other network measures can be described in terms of $\mathbf{A}$. The *degree* of a node in a network is defined as the number of edges connected to it. Here we denote the degree of node i by k_i. The degree of an undirected network of order n can be written in terms of $\mathbf{A}$ thus

$$k_i = \sum_{j=1}^{n} A_{ij}. \tag{2.3}$$

Since every link in an undirected network has two ends, if there are m links there must be $2m$ ends. This is also equal to the sum of the degrees of all the nodes meaning

$$2m = \sum_{i=1}^{n} k_i, \tag{2.4}$$

which can be rewritten as

$$m = \frac{1}{2}\sum_{i=1}^{n} k_i = \frac{1}{2}\sum A_{ij}. \tag{2.5}$$

The mean degree c of a node in an undirected network is

$$c = \frac{1}{n}\sum_{i=1}^{n} k_i, \tag{2.6}$$

which, when combined with equation (2.3), gives

$$c = \frac{2m}{n}. \tag{2.7}$$

Applying this to the example network $m = 8$ and $n = 6$,

$$c = \frac{2(8)}{6} = \frac{8}{3}. \tag{2.8}$$

The maximum possible connections available to a network (with $n \geq 2$) is $\frac{1}{2}n(n-1)$. This can be used to calculate the density

$$\rho = \frac{2m}{n(n-1)} = \frac{c}{n-1}. \tag{2.9}$$

The range of ρ is strictly $0 \leq \rho \leq 1$. Applying this to the example network (Figure 2.6) gives a density of $\rho = 0.2667$. Another measure of networks is the distance between any two nodes. The path is a collection of links mapping the route between the nodes. There can be a number of paths between two nodes. For example Figure 2.7 shows two different paths between node 0 and node 4. Calculating the number of paths of length r on a network can be achieved using A_{ij}. Since $A_{ij} = 1$ for links between nodes i and j; it follows that the product $A_{ik}A_{kj}$ is 1 if there is a path length of 2 from j to i via k. The total number N_{ij}^2 of paths via any other node is

$$N_{ij}^2 = \sum_{k=1}^{n} A_{ik}A_{kj} = \left[\mathbf{A^2}\right]_{ij}, \tag{2.10}$$

where $[\ldots]_{ij}$ denotes the ijth element of the matrix. This can be generalised to r,

$$N_{ij}^r = [\mathbf{A^r}]_{ij}. \tag{2.11}$$

The mean path length simple averages the path length in the network. More importantly the shortest path length (also called the geodesic path) between nodes i and j is the smallest value of r such that $[\mathbf{A^r}]_{ij} > 0$. The degree distribution (p_k) of the example network can be displayed as a histogram (Figure 2.8). The fraction of n nodes that have degree k can be calculated giving:

$$p_0 = 0,\ p_1 = \frac{1}{6},\ p_2 = \frac{1}{6},\ p_3 = \frac{3}{6},\ p_4 = \frac{1}{6}, \tag{2.12}$$

with $p_k = 0$ for $k > 4$. The value p_k can also be considered a probability of a randomly selected node having degree k; thus $\sum_{k=0}^{n} p_k = 1$. The total number of nodes with a given k can be calculated from np_k or can be read from the histogram.

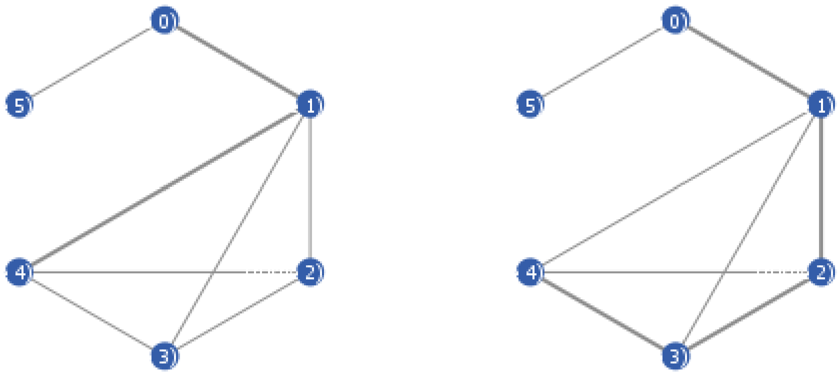


Figure 2.7: A comparison of path lengths between the same two nodes: 2 (left) and 4 (right)

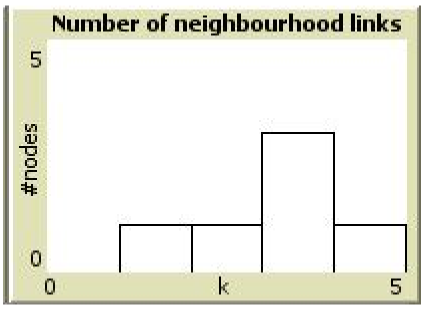


Figure 2.8: The degree distribution of the example network.

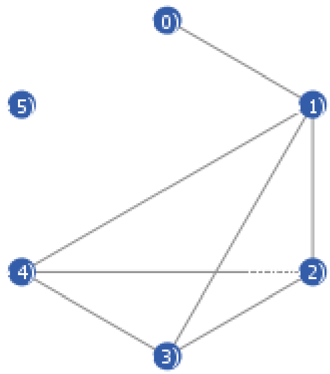


Figure 2.9: The network with $p_0 = 1$ means it now has two components

Although in the example network $p_0 = 0$, by removing the link between nodes 0 and 5 it is possible to have a network where $p_0 \neq 0$ (Figure 2.9). The resulting network now has two components, with

$$p_0 = \frac{1}{6}, \quad p_1 = \frac{1}{6}, \quad p_2 = 0, \quad p_3 = \frac{3}{6}, \quad p_4 = \frac{1}{6}, \tag{2.13}$$

with $p_k = 0$ for $k > 4$.

Networks are adept at modelling social systems, and are one of the few techniques already well used in the Social sciences. Social networks are considered self-organising and emergent [62], in other words complex. However they are difficult to reproduce in simulations, with most social network simulations using some variation of a random network. Recent work by Hamill and Gilbert [45] develops a social circle network model. The authours argue, that standard network models do not fit well with sociological observations of social circles. Key aspects of large social networks are the differing sizes of personal networks, limits to those sizes, high clustering, positive assortativity of degree of connectivity, and low density. Drawing on the metaphor of social circles, they define a personal network with five different strength of connections, from strongest to weakest. The stronger connections are fewer in number (under the, not unreasonable, assumption stronger ties take more ‘time and effort’ to maintain). By introducing a ‘social reach’, the agents in the network are limited in the number of social ties they can achieve. The networks then evolve connections based on these parameters. To do this 1000 agents are simulated on an unbounded grid of just under 100,000 cells. The agents then move around the grid making (and breaking) connections. Results are the average of 30 runs. The networks that are produced have characteristics of ‘real-world’ social networks. They vary in size, depending on the ‘social reach’ of individuals, which also limits the size of the network. Overlapping social reaches cause high clustering, whilst the movement of agents makes the networks dynamic. The social reach parameter seems to be the key to the development of the networks. Whilst this is something that could lead to criticism of simplicity, the authours note that their method simply "create[s] agent-based models that represent empirical social networks with greater veracity than the standard network models" [45, p91]. Certainly this is a useful method for developing 'non-standard' networks, and further applications are likely. It is interesting to note, the method the authours use to develop the network is Agent Based Modelling, which is one of the most used methods in Social simulation.

2.5 Agent Based Modelling

Agent (or Individual) Based Modelling (ABM) is the use of a multi-agent environment architecture to model a system. The architecture is able to overcome some of the limitations of previous techniques, by directly modelling individuals, and their interactions, in an environment. The agents are an abstract representation of the individual members of the population, and the environment they interact with is an abstraction of their real world environment. Therefore, modelling is done at a local scale, with the agent and its interactions with the environment being modelled, rather than attempting to model the population as a whole. It is defined by Wooldrige and Jennings [106] as

> "An *agent* is a computer system that is *situated* in an *environment* and that is capable of *autonomous action* in that environment to achieve its set objectives."

According to Gálan et al. [39], with agent-based modelling, the entities of the system are represented explicitly and individually (but not necessarily accurately) in the model. The limits of the entities in the target system, correspond to the limits of the agents in the model; and the interactions between entities, correspond to the interactions of the agents in the model. This abstract impression of an agent, means that it lends itself well to modelling a number of different phenomena. Since the modelling of a system can now be done at a local scale, it is believed that valid results will emerge from these individual processes.

There are those that argue agent models are non-mathematical and non-deductive, claiming they lack the mathematical rigour of equation based analysis. Epstein [29] counters these claims, arguing that every agent is a computer program and, as such, is computable by a Turing machine (Turing computable). Since for every Turing machine there is a unique corresponding, and equivalent, partial recursive function [47], in principle, one can cast any agent based computational model as an explicit set of recursive functions. Although, in practice, these might be extremely complex and difficult to interpret they surely exist. This is a powerful counter to the arguments that agent-based models are non-mathematical. As Epstein points out, the non-deductiveness follows since recursive functions are computed deterministically from initial values. Given the nth (including the initial) state the nth+1st state is computable, in a strictly mechanical and deterministic way, by recursion.

According to Epstein [31], Agent Based Modelling was first used by Schelling in his 'Dynamical Models of Segregation' [74] (although Schelling used different terminology). During the first half of the 1990s researchers at the Santa Fe Institute, led by Joshua Epstein and Robert Axtell, used agent architecture techniques and developed Sugarscape, an agent based simulation, using a generative social science model [31]. The Santa Fe Institute has long argued that ABMs are an excellent tool for modelling social phenomena, such as segregation. This is in no small part due to their suitability for modelling complex agents. ABMs have been able to show how simple, and predictable, local interactions can generate familiar global patterns, such as the diffusion of information [31], emergence of norms [71], wealth aggregation [31], segregation of populations [74] or participation in collective action [57]. Epstein [29] talks of a generative social science and growing artificial societies from the bottom up claiming "if you didn't grow it, you didn't explain its emergence." Macy [57] gives strong support to the generative views of Epstein, believing that ABMs offer theoretical leverage, where the global patterns of interest can be seen to be more than, the aggregation of individual attributes. Macy argues that emergent behaviours cannot be understood without a bottom up dynamical model of the micro processes from individuals [57]. The simulation of these individuals combines to give, what Sawyer calls, artificial societies, which he defines as "a set of autonomous agents operating in parallel and communicating with each other" [73, p2]. Sawyer argues these artificial societies will lead to an understanding of the mechanics of micro-to-macro emergence, macro-to-micro causation and "the dialectic between social emergence and social causation" [73, p8]. Macy hopes that ABMs could offer an alternative approach for sociologists (and all social scientists), who have often modelled social processes as interactions among variables [57]. Certainly ABMs seem to offer an excellent way to model social systems. Schelling [74] used an ABM consisting of coins and a checkers board to explore the effects of individual preference on global segregation (the experiment is explored in full in section 3). His conclusions generated huge interest and some controversy with sociologists such as Massey [58] and Yinger [107] criticising its simplistic nature. Even so, Schelling's models of segregation have become well known within the social simulation community, precisely because they provide alternative modes of exploring issues, more commonly investigated using traditional statistical analysis. The apparent ease of modelling using ABMs, most notably by Epstein [29] and the Santa Fe Institute, but also by Gilbert [43], Sawyer [73] and others, has led to an explosion of ABMs modelling social interaction. This is well documented by both Railsback [69] and Nikolai and Madey [64] who identify over 50 current ABM platforms. However for all these platforms there is no universal approach to this technique. Epstein believes a framework will emerge over time, as these many approaches merge and are refined [30].

In the mean time, Laurie and Jaggie [53] use ABM techniques to explore some of the effects of changing the values of parameters selected by Schelling. They suggest most work on Schelling's models has focused on the preferences (p) of agents (defined in Section **??** as τ). Although Laurie and Jaggie use p, their main focus is on the neighbourhood of agents, up to a distance of R, calling this parameter an agent's vision. The environment consists of an $N \times N$ square array, with periodic boundary conditions. This creates a society on an edgeless torus rather than a square grid which, they argue, suppresses possible boundary effects. They then propose two more parameters: c for concentration of the minority race and v, for the concentration of vacant residences. The system is initialised with $N = 50$, whilst v is randomly initialised so that $(1-v)N^2$ sites are occupied by $c(1-v)N^2$ Blacks, and $(1-c)(1-v)N^2$ Whites. Since they are only interested in the effect of R, they set $c = 0.5$, so that the number of Blacks is equal to the number of Whites. At each iteration an agent is chosen at random for 'evaluation'. The agent checks its R-neighbourhood and makes a movement decision based on the ratio of the neighbourhood. If the ratio of an agent's own type is greater or equal to p the agent is 'satisfied' and will not move. However if the ratio is lower the agent makes a series of attempts to move. Selecting a random vacant site the agent calculates the ratio and, if its satisfaction is increased, moves. If the satisfaction is not increased (i.e is the same or lower) the agent randomly selects a different vacant site repeating the process a maximum of vN^2 times before admitting defeat and staying put. To measure the degree of segregation within the system, Laurie and Jaggie define an "ensemble averaged, von Neumann segregation coefficient at equilibrium" [53, p2693]. This value S is defined as;

$$S = \frac{1}{(1-v)N^2}\left[\sum_{j,white}\frac{(f_j - f_w)}{(1-f_w)} + \sum_{k,black}\frac{(f_k - f_b)}{(1-f_b)}\right]$$

where $f_w(c)$ and $f_b(c)$ represents the expected fraction of white or black neighbours respectively from a random initial configuration. Their results show that for $p < 0.4$ the population does indeed reach a stable equilibrium with $S < 0.5$ (i.e. more integrated than segregated). Going further their model shows that when $p = 0.3$ and $R = 5$, $S = 0.03 \pm 0.03$ suggesting the possibility of a completely integrated stable state. Laurie and Jaggie argue that there is "a large region of the parameter space (p, R), particularly for moderate values of $R(2 \leq R \leq 7)$, where integrated communities remain stable for arbitrarily long times." They claim that, once R is expanded from the myopic level of Schelling to modest levels $3 \leq R \leq 5$, non-segregated stable communities form. This happens even when preference p is non-zero and "quite substantive" [53, p2691].

Whilst the results are interesting, their hope that it could offer insights to policy makers is difficult to support. Firstly their environment has no real relation to reality. With no geographic or economic factors, the model is just too abstract to be able to claim any relation to reality. Laurie and Jaggie's suggestion seems not to add anything to defend the criticisms levelled at Schelling's models about their simplistic nature (discussed in section 3). Indeed one would argue their suggestion of policy implementation, coming without consultation of any sociological experts (when both are physicists), compounds the problems agent based modellers have when trying to model social systems, that of credibility amongst the sociological community. Additionally, their use of two equal populations completely removes any ideas of minorities which Schelling was attempting to address. However they highlight an important factor in the value of p. They report that in a number of studies (although they only cite Epstein and Axtell [31]) $p = 0.5$ is considered a 'colour-blind' value. However, they rightly point out that far from being colour-blind an agent with $p = 0.5$ will never be 'happy' in a minority and will actively seek to leave any neighbourhood within which they are not at least equal.

The success of ABMs in modelling social interactions, has led to the field of Computational Social Science. This fast growing field can be traced back to Epstein's book 'Generative social science' [29]. The field combines social models and computer simulation, mainly through ABMs, and attempts to give social scientists access to the power of simulation. Although other techniques are applicable (for example network models), ABMs are by far the most common. This is could easily be attributed to the abstract nature of ABMs, most models can be described as ABMs. This, almost overwhelming, use of a single modelling technique, limits the ability of the field to offer different perspectives of the same problem. Finally, there is a noticeable lack of discussion of the environment. Although it has been fifteen years since Beer [14] stated, [emphasis added]:

> "we must learn to think of an agent as containing only a latent potential to engage in appropriate patterns of interaction. It is *only when coupled with a suitable environment* that this potential is actually realized through the agent's behavior in that environment."

Since then there has been relatively little progress in incorporating spatial factors in ABMs. The focus of ABMs on agents is understandable and, once again, Beer is correct in asserting that "an agent's behavior properly resides only in the dynamics of the coupled system and not in the individual dynamics of either" [14]. Thus, for the power of ABMs to be truly exploited there is a need for improvement in the models of the environment. The rigid conformity of grid based systems is slowly being reduced, although ideas of irregular grids still cling to abstract ideas of space. Surprisingly, a way forward has been suggested by the video game industry. Their consumers demand highly realistic and dynamic environments that are coupled with intelligent agents. This demand has led to the implementation of Geographical Information Systems into ABMs and offers the possibility of building models with much more complex spatial environments.

2.6 Geographic Information Systems

Information systems are a useful tool to manage knowledge by making it easy to organize, store, access, manipulate and synthesise, as well as applying the knowledge to a problem. Geographic Information Systems (GISs) are a form of information system where the knowledge is geographical. These GISs are tools that are often developed in a task specific way, so that two GISs are often incompatible. GISs were initially developed in Canada in the 1960s by Roger Tomlinson and colleagues for the Canadian Land Registry. These systems were able to hold information about a spatial location and their inital success meant that within 5 years a GIS lab has opened in Harvard [56]. Although initially developed by town planners, recent developments have attempted to combine GISs with Agent Based Models. O'Sullivan [65] believes GISs form an important part of modelling complex systems and argues they are a fundamental tool for modelling spatially explicit agent based models of social systems. A GIS is able to define multiple scales, from streets to neighbourhoods to areas to cities. Rather than having a static grid environment, GIS can offer the opportunity to explore dynamical evolving environments. Using a real world example of Yaffo (an area in Tel Aviv, Israel), Benenson *et al.* [15] implement a GIS environment (Figure 2.10) into an ABM of Schelling's segregation model. The selection of the area is based on two main factors. Firstly, there is 50 years of empirical data about the makeup of the population; secondly, the street network has remained relatively unchanged over the timescale with residential construction limited. Within the environment a population of 30,000 agents representing Arabs and Jews is randomly distributed with a ratio of 1:2. This ratio is selected on the basis of census information from 1955. Residencies are split into two types, oriental and modern block buildings. Agents' preferences were asymmetrical in relation to each other, Jews preferred to

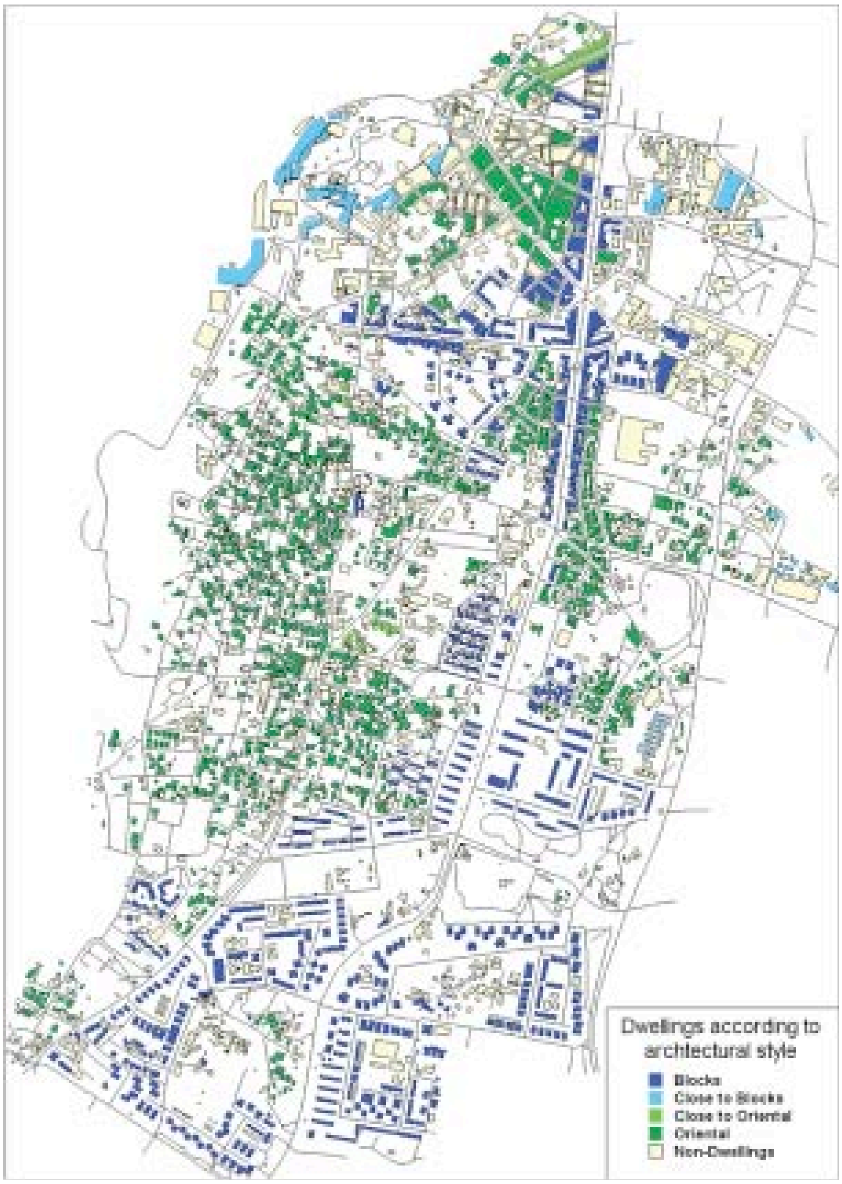


Figure 2.10: A GIS representation of Yaffo from [16]

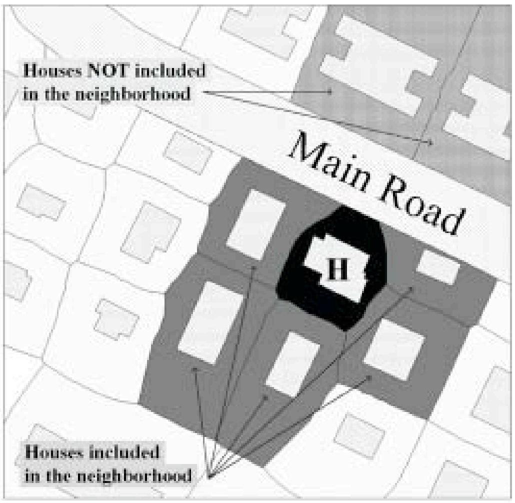


Figure 2.11: An example neighbourhood from [16]

live with Jews, whilst Arabs have no preference. A neighbourhood is defined as a group of neighbouring houses whose Voronoi polygons share an edge (Figure 2.11).

If an agent is 'unsatisfied' they are given K opportunities to move. Agents assess K options and places them into an order based on a utility u_i with $0 \leq u_i \leq 1$ for $i = 1, 2, \ldots, K$ before making a choice. If an option is available and the utility exceeds an agent's current utility, the agent moves based on a probability $p_i = p(u_i)$ (i.e. $p(0) = 0, p(1) = 1$). If an agent does not move they check the next possibility until either they move or reach the end of the list. Using this simple model, Benenson *et al.* are able to produce results that are comparable to empirical results [16]. From this, they believe that it is possible to, not only model urban social change, but also to model it accurately. However, there are a number of issues with the model that they present. Most importantly is the classification of two types of housing and their respective desirability for the two populations. Although they cite empirical research that suggests this preference is a real world phenomenon, the addition of this to the agents' utility model certainly drives the populations into the dwellings that have been pre-determined. Additionally, the idea that a house situated across a road from another house is not considered part of an individual's neighbourhood is unrealistic. It would be more realistic to have a diffused model of preference, so that the houses could be considered by agents, but are less important than connected neighbours. Furthermore there is little explanation about the passage of time. The number of moves allowed per year is not mentioned in the paper. Certainly there is no indication how this timescale is broken down, apart from suggesting a total timeframe of 50 years. Still the experiment is a novel application of a GIS to an ABM, and should be an important tool in future research.

More recently, ideas of Participatory GISs have been advocated. Participatory GIS looks to utilise the knowledge of local communities to enhance the applications of GISs. It is hoped that by combining GISs and Participatory Learning and Action PGISs will increase the usefulness of GISs [26]. By inviting this input it is hoped the previously static nature of GISs can become more dynamic and responsive to the needs of users. The practice integrates several tools and methods whilst often relying on the combination of expert skills with socially differentiated local knowledge [26]. It promotes interactive participation of stakeholders in generating and managing spatial information and it uses information about specific landscapes to facilitate broadly-based decision making processes that support effective communication and community advocacy [26]. However the approach is limited to technologically confident participants, meaning some important stakeholders could be missed.

2.7 Summary

One of the problems of dealing with complex systems is that complexity is not boolean. Rather, complexity can be considered a scale from simple complex systems, such as reacting particles, to highly complex systems, such as social systems. Additionally the complexity can be in both the environment and the components within the environment. The problem is probably best described by Herb Simon's 'situated ant' [83]. An ant following a random walk on a beach produces a complex path. An observer watching might be amazed at this complexity created by the ant. However, Simon suggests, the complexity of the environment is creating the complex path rather than the ant. The analogy suggests complexity of the environment is as important as the complexity of the individual.

From the previous sections, it would appear that complex systems can be considered (at least) an individual situated within an interacting environment. By creating a matrix based on the complexities of *both* the environment and the individuals one should be able to classify the system being studied accordingly. Taking the environment as the horizontal axis, it is clear there are a number of ways to represent the environment. Aspatial environments are probably the simplest since their is no spatial representation, the environment is considered just the mixture of individuals. Network space is a graphical representation of connections (often called vertices) between individuals (nodes). Distance is considered in terms of the number of nodes between two individual nodes, rather than Euclidean distance along the links. Structured space has a notion of distance between nodes but it is not a metric as it does not satisfy the triangle inequality (*i.e.*, for the distance between two nodes, A and C, is not necessarily less than or equal to the distance between A and B plus B and C).

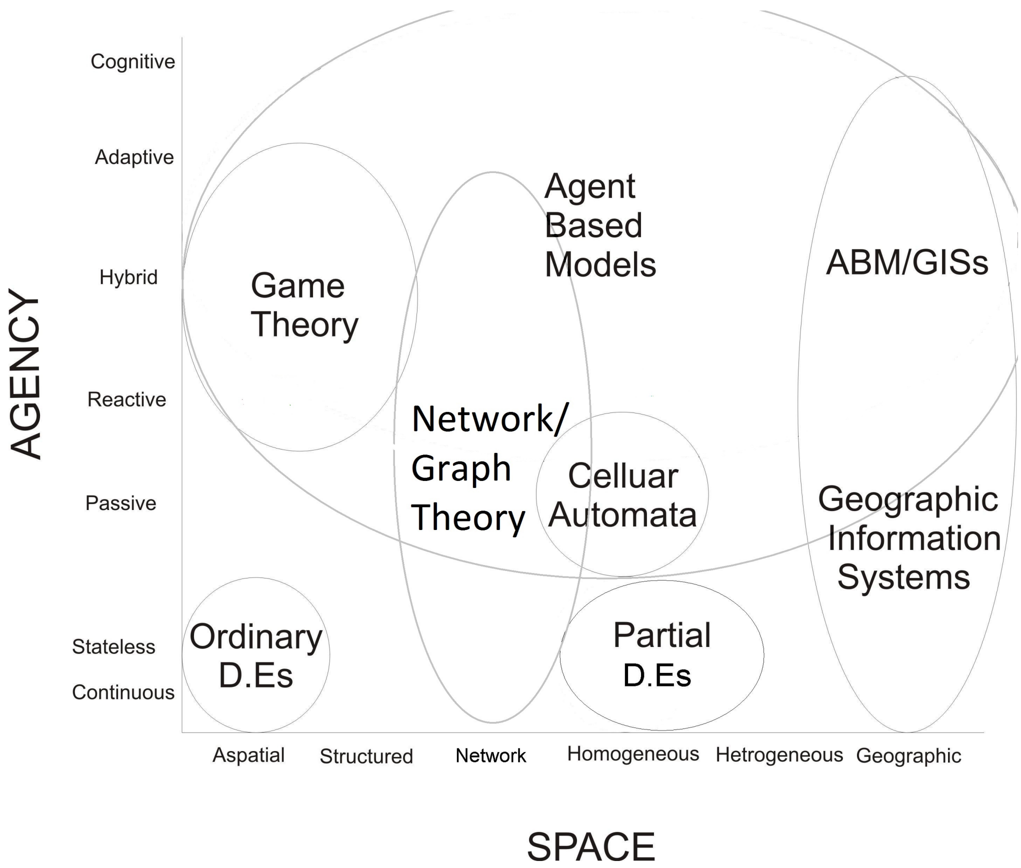


Figure 2.12: The Social Simulation Matrix. D.E = Differential Equation. As space and agency complexity increase, different modelling techniques are needed to capture the increased complexity. The matrix is an attempt to map techniques in modelling social systems to the complexity of the object being modelled. The matrix highlights the breath of modelling covered by Agent Based Models.

Moving away from these abstract notions we come to simple homogeneous space. Here the environment is a uniform collection of Euclidian space incorporating ideas of distance and location. Introducing variation in the space takes us to heterogeneous space. Now different spatial locations can have different attributes. Increasing the heterogeneity of the environment eventually brings us to geographical space which can be considered an almost one to one mapping with a 'real-world' environment.

Individuals within the environments can be scaled according to agency. By this we mean an ability to act independently and make choices based on available information. The simplest form of agency is a population level, statelessness form. Individual components are indistinguishable and, therefore, treated as a continuum rather than as single components. Identifying individual components allows us to introduce ideas of agency properly, the simplest form of which is passive components. As the name suggests these components are unable to respond to any changes in their environment and continue with their set goal irrespective of external influence. More complex reactive components are able to react to environmental changes to achieve goals. This reaction is based on a hierarchical set of behavioural rules. Hybrid components introduce a subsystem responsible for abstract planning and decision making. Adaptive components are similar to hybrids but have an important addition, namely they are able to learn from their environment. Whilst reactive and hybrid components have a knowledge about their environment, adaptive components have additional knowledge about their own structure and evolutionary capacity. This knowledge and learning ability allows components to adapt their behaviours to reach their goal. Finally cognitive components are able to create a symbolic internalised model of their environment. From this model they are able to reason and plan strategies to achieve their goal.

By plotting these two aspects of a system we can produce the social simulation matrix in Figure 2.12. The matrix can be used to guide the process of selecting a modelling tool for the task at hand. As complex systems are multi-faceted, the toolset needed for analysis can be compared to the Swiss army knife. The very nature of a complex system requires many tools to process the huge amount of information. With so many tools available to model social systems it is useful to map the available techniques onto the social simulation matrix. The development of these (and other) techniques using computational modelling and simulation, has led to the birth of Computational Social Science [29]. One of the most well used models in the field is Schelling's 'Dynamic Models of Segregation', which is now introduced in detail.

Chapter 3

Schelling's models of Segregation

Schelling's models of segregation have been the focus of a number of studies using a variety of different techniques. In his ground breaking paper [74], Schelling used ABMs to explore ideas of segregation in heterogeneous populations. The models he developed used simplified ideas of social interaction. As has been mentioned, this has been described as the first ABM [30]. It has also been called the first simulation of an *artificial* society [73]. Although described as 'elegant' [28], Schelling's models are at times confused (for example, Axelrod talks of Schelling's Bounded Neighbourhood Model but describes the Spatial Proximity Model [10]) and closer analysis reveals intricacies (such as the movement of the agents) that have been overlooked. Schelling actually describes two distinct models, a Spatial Proximity Model [74, p149] and a Bounded Neighbourhood Model [74, p167]. Finally he combines aspects of the two into a Tipping Model [74, p181]. His models are now described in detail.

3.1 Spatial Proximity Model

The Spatial Proximity Model explores the aggregate/global effects of two groups of characters making decisions based on a local neighbourhood N for each character. Schelling created a one dimensional environment with two types of characters, equal in number and randomly arranged along the environment. The environment was Schelling's typewriter which defined the limits of 70 characters per line. Each character has a location, a thresholded 'happiness' about the makeup of their own neighbourhood N and a movement rule. An example of the system can be seen in Figure 3.1. For characters situated a distance of at least n from the boundary, Schelling chooses the neighbourhood N to contain $2n$ characters, n on the left and n on the right. For characters closer to the boundary, say $m < n$ units, he considers a reduced

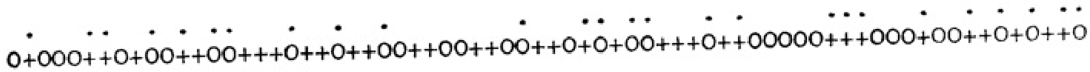


Figure 3.1: An example initial configuration of Schelling's one dimensional Spatial Proximity Model [74]. Marks above characters indicate 'unhappiness'.

neighbourhood with $m+n$ characters. An iterative process described in algorithm 1 is then implemented. For a general neighbourhood a rule is applied, such that, if within N, less than n are of the same type as the character, it is deemed 'unhappy'. For boundary conditions the rule is modified, such that, if less than $\frac{m+n}{2}$ are the same type a character is 'unhappy'. To alleviate their 'unhappiness' a movement rule is defined, such that, an 'unhappy' character can move to the nearest position that would satisfy its 'happiness'. If a 'happy' position is equidistant, characters move to the right.

Algorithm 1 Spatial Proximity Model

```
while System not at equilibrium do
  for each character do
    calculate 'happiness'
  end for
  for i = 1 to 70 do
    if char i 'unhappy' then
      movement rule
    end if
  end for
end while
```

Algorithm 2 Calculate 'happiness'

```
count agents in N
if number of own type < n then
  'unhappy'
else
  'happy'
end if
```

Characters are considered in turn from left to right. If a movement changes a character from 'happy' to 'unhappy', the character must wait until the next iteration to move. If a character becomes 'happy' they no longer move. After each iteration, if any characters are 'unhappy', the process repeats until all agents are 'happy' (equilibrium) and the while loop is terminated. To give an illustrative example, if we set $n = 2$ the neighbourhood of a '+' would

Algorithm 3 Movement rule

if ‘unhappy’ **then**
  move to nearest ‘happy’ position
end if

be ‘$\underline{\mathsf{oo}}\mathsf{+}\underline{\mathsf{oo}}$’. Applying the rules to an initial state and highlighting (a lá Schelling) ‘unhappy’ characters produces:

$$\mathsf{oo}\underline{\acute{+}}\mathsf{o}\acute{\mathsf{o}}\acute{+}+\acute{\mathsf{o}}+\acute{+}\acute{\mathsf{o}}\acute{\mathsf{o}}\acute{+}\acute{+}\acute{\mathsf{o}}$$

Taking each agent in turn from left to right, the first agent move (highlighted $\underline{+}$ moves 2 places right) would produce:

$$\mathsf{oooo}\underline{+}++\underline{\acute{\mathsf{o}}}++\acute{\mathsf{o}}\acute{\mathsf{o}}\acute{+}\acute{+}\acute{\mathsf{o}}$$

Since a number of ‘unhappy’ characters have been made ‘happy’ by the move they chose not to move and the next character move gives:

$$\mathsf{oooo}+++++\underline{\mathsf{o}}\mathsf{oo}\underline{\acute{+}}\acute{+}\acute{\mathsf{o}}$$

$$\mathsf{oooo}+++++\underline{+}\mathsf{ooo}\underline{\acute{+}}\mathsf{o}$$

and finally:

$$\mathsf{oooo}++++++\underline{+}\mathsf{oooo}$$

which is an equilibrium (i.e. all characters are at least satisfied with their neighbours).

If a previously ‘happy’ character is made ‘unhappy’ by another’s move, this new ‘unhappy’ character must wait until all the other characters have moved before they can themselves move. So for example:

$$+\underline{\acute{+}}\mathsf{oooo}++++\mathsf{oooo}\acute{+}+$$

the movement of the first $\underline{\acute{+}}$ leads to:

+oooo+++++oooo++

which creates a newly ‘unhappy’ character (the leftmost +). This character must wait for the iteration to complete before itself moving.

Schelling concludes that, with all things being equal, different random sequences of 70 characters with $n = 4$ “... yield[s] from about five groupings with an average of 14 members to about seven or eight groupings with an average of 9 or 10, six being the modal number of groups and 12 the modal size.”

To highlight this finding two different initial configurations with the same ruleset are explored below. If a character is equidistant from a ‘happy’ position, it will move to the right. The first configuration achieves equilibrium after one iteration:

o+++++oo++ooo++oo+++oooo++++ooo+++++oooooo+++oooo++++oooooo++o++ooo+++

+++++++oooooooooooo++++++++++++++ooooooooooooo+++++++oooooooooo+++++++

as does the second:

++o++o++oo+++ooo++++ooo++oooo+++ooooo++++oo++++oo+oo+ooo++++ooo++ooo+o

+++++++++ooooooo++++++oooooooooooo++++++++++++oooooooooo++++++++oooooo

A third configuration has two runs to explore the effect of the movement rule. The first is from Schelling’s movement rule:

oo+++ooo+++oo+++oo+++oooo+++oooo+++ooo++ooo+++ooo+o+o+++ooo++++oooo+++

+ooooooo+++++++++++ooooooooo+++o+++o++oooooooooo++++++++++++++ooooooo+

ooooooo++++++++++++oooooooooo++++++++ooooooooooo+++++++++++++++ooooooo

and the second from modifying a movement rule to consider the characters in a random order (rather than the sequential left to right order):

oo+++ooo+++oo+++oo+++oooo+++oooo+++ooo++ooo+++ooo+o+o+++ooo++++oooo+++

ooooooo++++++++++++++ooooooooooo++++++oooooooooo++++++++++++++ooooooo+

ooooooo++++++++++++++ooooooooooo++++++oooooooooo+++++++++++++++ooooooo

As these results show, the movement rule of left to right or random does not produce qualitatively different results; indeed the final states are identical.

Schelling's model produces a higher degree of segregation than he expected. He argues that because smaller groupings of regular alternation of characters (as well as alternating pairs) would also be equilibria their non-appearance was "striking" [74]. For example, the infinite configurations:

...o+o+o+o+o+...
...oo++oo++oo++...

are equilibria under Schelling's rules. However, this only holds for even n. If the neighbourhood for n is odd all characters would be 'unhappy'. This complete switch from only a minor parameter change suggests the system is actually unstable. Additionally, the introduction of boundary conditions would always lead to 'unhappy' characters at the boundary. The instability of these systems is reduced as the size of the clusters increases. Intuitively, one can see that the characters are acting with "complete knowledge", insofar as, when a character moves, it is always towards a neighbourhood with a higher percentage of its own type than the one it currently occupies. The aggregate result of these conditions means that any cluster of the same characters of size greater than $2n + 1$ cannot be split and thus will only grow in size. These emergent phenomena combine to produce the "striking" results of Schelling but can actually be deduced. These larger groupings are substantially more stable than the alternating and pairwise equilibria. Certainly, with the chosen 'happiness' thresholded (i.e. 50%), alternating characters would only be 'happy' with a certain neighbourhood size. Additionally there are situations where the system will never reach an equilibrium. If we take $n = 2$ the following system:

ooo++

will cycle between two configurations as the two '+' characters continually swap positions. From this we can define an axiom of the system; to reach an equilibrium there must be at least $n + 1$ characters of the same type. Schelling himself didn't have the tools available to study this phenomena and subsequent work has tended to ignore it, instead concentrating on the two dimensional models. However, as we have been able to demonstrate with these simple examples, the system will tend to a highly segregated configuration.

From his Spatial Proximity Model, Schelling identified the following five parameters available for variation:

- Neighbourhood size;
- Ratio of populations;
- Rules governing movement;
- Original configuration;
- Demanded percentage of own type;

Neighbourhood size is one of the obvious parameters to explore. Schelling reduces n by 1 (from 4 to 3) and reports that this produces an average of 7 to 8 characters per cluster. Schelling is unclear as to whether this average is the mean, median or mode, but we will assume this is the modal value, consistent with his previous workings. Although not an overly surprising result, Schelling did discover an interesting arrangement of the smallest possible cluster sizes, that is . . . `oo+o++o+oo+o++`. . . which is only stable if the ends repeat indefinitely (*i.e.*, forms a recurring symmetric pattern). Smaller cluster sizes of alternating characters are completely unstable owing to the reasons discussed earlier.

Schelling investigates whether the ratio of the populations has any effects, and, in an attempt at fairness, randomly removes characters from one of the populations. With this method, 17 of 35 'o's are removed. As the inequality between the populations increases, so does the degree of segregation. Although Schelling fails to measure this property with mathematical rigour, it becomes clear that as the number of a population falls, so does the probability of the existence of a stable cluster the increasing the degree of segregation of the population. It is obvious that if one population falls below $n + 1$ the system will never reach equilibrium. Similarly until the population reaches $2(n+1)$ the maximum number of clusters will be 1.

Schelling theorizes about restricted movement, in which a character who cannot move to somewhere with half their own type, moves to somewhere with $\frac{3}{8}$ instead. He suggests that the neighbourhoods move to equilibrium at a quicker rate (although he is unclear how much quicker) which seems reasonable.

3.1.1 Extended Spatial Proximity Model

Schelling explores what one might consider the most influential of the parameters, namely a character's demand for their own type within their neighbourhood. Although he studied these effects in detail, Schelling felt the one dimensional environment was too restrictive and, instead, considered a two dimensional grid. The setup introduced ideas of empty space to "make search and satisfaction possible" [76]. Although the environment has changed,

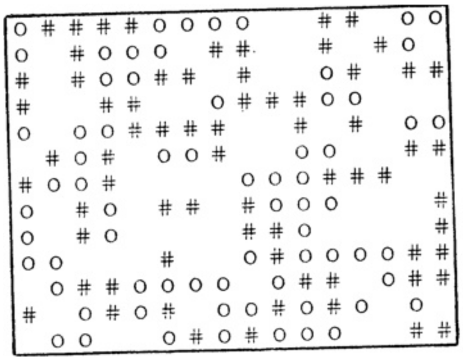


Figure 3.2: An example initial configuration of Schelling's two dimensional Spatial Proximity Model [74].

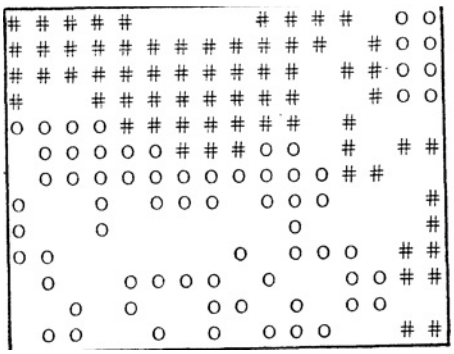


Figure 3.3: Final configuration after left to right, top to bottom movement rule [74].

Schelling attempts consistency by keeping the ruleset the same. Thus, 138 characters are equally split (39 # and 39 o) and randomly distributed across the environment. The remaining spaces are left blank to allow movement of characters. The definition of neighbourhood is changed to a Moore neighbourhood (i.e. the eight cells surrounding the central cell in a 3x3 group). Characters on a boundary edge only count the cells within the boundary as their neighbourhood. 'Unhappy' characters move to the closest position that satisfies their desired neighbourhood makeup (i.e. 50% of their own type). If two positions are equidistant one is chosen at random.The choice of character to move is chosen from left to right, top to bottom. However, Schelling admits that "no exact rule for order of movement has been strictly adhered to" [74]. To highlight the effect of the movement rule, Schelling offers the initial system set up (Figure 3.2), which produces the final configuration (Figure 3.3).

Schelling argues this is "too striking to need comment" [74]. But by doing so he misses an important point: the segregation is still driven by the same rules as the one dimensional

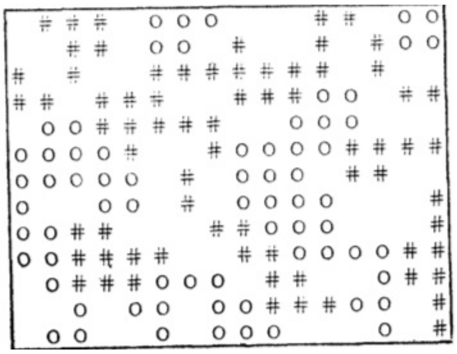


Figure 3.4: Final configuration after ‘random’ movement rule [74, p157].

model. Thus characters are still unable to start their own group and instead must just join a larger group of their own type. By changing the movement rule so that ‘unhappy’ characters are selected and moved in a random order, Schelling produced a different final configuration (Figure 3.4). At first glance Figure 3.4 does not seem segregated. Certainly Schelling has to highlight the boundaries to emphasise his point. However, by using a neighbour count on the initial and final configurations Schelling is able to argue the system is segregated. His analysis suggests in the initial configuration the average of like neighbours was 53% for zeros, and 46% hashes. The analysis of the final configuration (Figure 3.4) showed both average values had jumped over 30%. Zeros were now in neighbourhoods that were 83% their own type, whist hashes were 80%. His analysis of Figure 3.3 showed the value was 90% for both with 66% having no neighbours of opposite colour at all [74]. So, Schelling argues, although the particular outcome will be highly dependent on the movement order the character of the outcome (i.e. the degree of segregation) will not. This result agrees with the findings from the one dimensional model experiments earlier. As in the one dimensional environment, the two dimensional system tends towards a more segregated setup as this is more stable. Although it is possible he missed some of the nuances garnered from the one dimensional model, the analysis of the two dimensional produced some quite unexpected results. By changing the demand of one population to seek minority neighbourhoods, Schelling expected the degree of segregation to reduce as a population became more tolerant. However he found it made no significant difference to the segregation [74]. If one population tolerance remains the same, their level of segregation will remain the same. Since there are only two populations overall the average degree of segregation should not change. The only change will be in the stability of groups of the more tolerant population, which will be higher.

An interesting point Schelling found was that if both populations sought integration, and one was a minority, the minority population became ‘shared out’ amongst the majority [74].

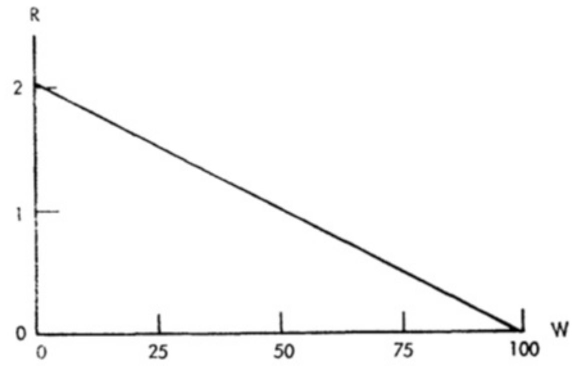


Figure 3.5: Schelling's Tolerance Schedule for the Bounded Neighbourhood Model setup [74]. R is the ratio that can be tolerated.

Thus the segregation of the populations remained. Schelling admits that the small sample of his analysis means there is little scope for generalisation. But, he argues, "comparisons among them . . . such as the effect of reducing or enlarging a minority, may be capable of some extension to that world" [74]. It is this proclamation of real world relevance (and others like it) that caused the harshest criticism of his model, the lack of socio-economic elements (see Yinger [107] and Massey [58]).

3.2 Bounded Neighbourhood Model

Having explored the effects of local preference, Schelling turns his attention to global preferences with a Bounded Neighbourhood Model. Whilst there have been a number of papers discussing Schelling's Spatial Proximity Model (commonly referred to as Schelling's segregation model) there is less work on his Bounded Neighbourhood Model. The confusion seems to be over the *spatial* aspects of the model. In the Bounded Neighbourhood Model, characters consider the proportions within the system (aspatial, Figure 2.12) rather than proportions within a local neighbourhood of the system (homogeneous, Figure 2.12). Although the jump from an individual level Spatial Proximity Model [74, p149] to a system level Bounded Neighbourhood Model [74, p167] is well defined in the paper, there are still confusions about the conventions of the two models. For example Axelrod discusses "Schelling's tipping model" [10, p16] when he actually means the segregation model. In the Bounded Neighbourhood Model, Schelling looks at global population flows into and out of a bounded neighbourhood. Characters are defined as black ($b \in B$) and white ($w \in W$) suggesting a more 'real world' application. Now, rather than identical individual tolerance, the tolerance of each individual is defined by a straight line cumulative distribution schedule plotted against R, the ratio that a character is willing to accept in the neighbourhood, with a maximum of $R = 2$ (*i.e.*, a ratio of 2:1) (Figure 3.5).

Thus, according to Schelling,

> "for each number of whites along the horizontal axis the number of blacks whose presence they will tolerate is equal to their own number times the corresponding ratio [on Figure 3.5]."

The schedule is applied at a population level. That is, for a population of $100W$; $25W$ can tolerate $37.5B$ (25×1.5), $50W$ can tolerate $50B$, $75W$ can tolerate $37.5B$ (75×0.5) and the most tolerant W is 'happy' with two Bs for each W. So long as the system configuration lies below the curve, the population is 'happy'. If the system lies above the line (i.e. $W = 100, R = 1$), there will be 'unhappy' members of the population. The tolerance schedule is applied to both populations B and W irrespective of whether they are inside or outside of the bounded neighbourhood. Thus, those outside the system will enter based on the current make up of the system population.

Algorithm 4 Bounded Neighbourhood Model Initialisation

Create population of m chars of type (b,w)
add $n < m$ chars to neighbourhood
apply tolerance to chars
execute System Dynamics

Algorithm 5 System Dynamics

```
while System not at equilibrium do
  for each char do
    calculate neighbourhood ratio
    calculate 'happiness'
  end for
  if char 'unhappy' and char tolerance == min tolerance then
    remove char
  end if
  for each char outside neighbourhood do
    if # chars in neighbourhood < max # chars and char 'happy' with neighbourhood
    configuration then
      add char
    end if
  end for
end while
```

The system is initialised (algorithm 4) with n characters of both types up to a specified maximum (100W and 50B in Figure 3.6) and set in motion (algorithm 5). The dynamics of entry and exit of the populations are described in Schelling's own words:

> "if all whites present in the area are content, and some outside would be content if they were inside, the former will stay and the latter will enter; and whites will continue to enter as long as all those present are content and some outside would be content if present. If not all whites present are content, some will leave; they will leave in order of their discontent, so that those remaining are the most tolerant; and when their number in relation to the number of blacks is such that the whites remaining are all content, no more of them leave. A similar rule governs entry and departure of blacks."

This description is unclear as the word "some" can have different interpretations, which could lead to different results. For example, there could be a difference between characters leaving en mass, rather than if they left one at a time. However, for most cases, the end result is the same; the difference lies in the number of iterations needed to reach equilibrium. Two

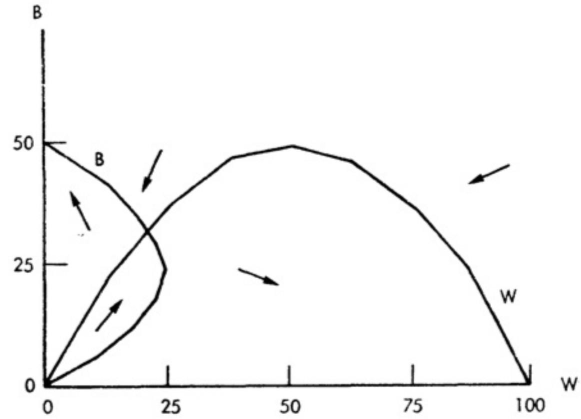


Figure 3.6: Schelling's Bounded Neighbourhood Model [74]. Schelling translates the straight line tolerance schedule in Figure 3.5 into 'population flows' with distinct parabolas for the two populations. The W population is represented on the x-axis and the B population on the y-axis.

approaches can be implemented by adding or subtracting a 'for' loop to contain the 'remove char' instruction.

Using his figures Schelling plots a graph (Figure 3.6) with a W population measured on the x axis and B on the y axis. Limiting the population of Bs to 50 and Ws to 100 Schelling manually plots the flows of the populations onto the graph to produce Figure 3.6. By applying his tolerance schedule to the initial populations of 100 W and 50 B the result would be 50 'unhappy' Ws and 49 'unhappy' Bs. This 'unhappiness' would cause a desire to exit and is shown by the arrow furthest right on Figure 3.6. It is assumed that 'some' of both populations leave at an equal rate (as indicated by the direction of the arrow), thus the ratio remains unchanged. As the numbers of both populations fall, the system configuration intersects the W parabola. This has two effects; firstly, all the Ws inside the system are 'happy', and, secondly, some Ws outside the system would be 'happy' in the system and begin to enter. The B population is still not 'happy', so some continue to leave, producing the arrow pointing right and downwards (increasing whites, decreasing blacks). It seems clear from this that the system is heading towards an equilibrium of 100 Ws and 0 B. Similarly an initial population of 50 Bs and 25 W produces the arrow pointing down and leftwards. This flow intersects the B parabola first and drives the system towards the equilibrium of 50 Bs and 0 W. The intersection of the two parabolas is an area Schelling calls a statically viable mixture of populations. It is characterised by increases in both populations up to the meeting point of the two curves at around (20,30). At this point, the entry of the next character becomes a defining characteristic of the system. Whichever type enters will drive

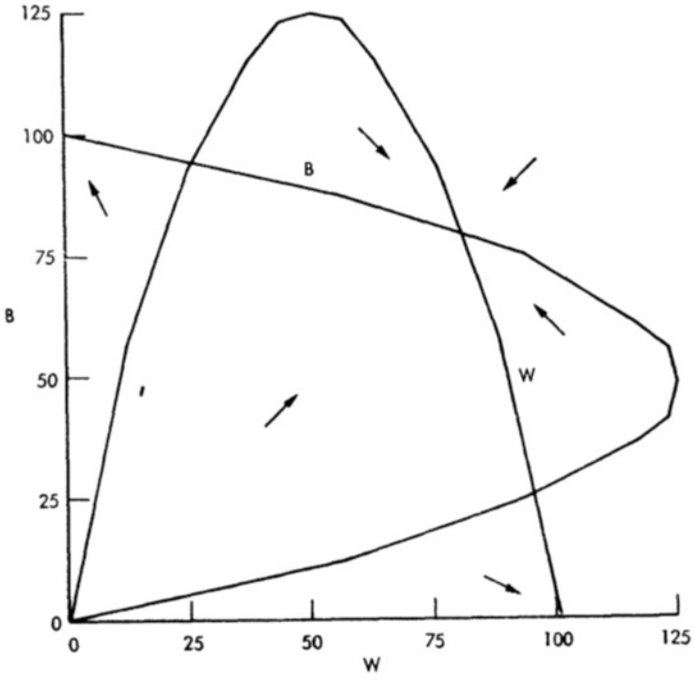


Figure 3.7: Schelling's Bounded Neighbourhood model setup with equal populations and 5:1 tolerance [74].

the system to their equilibrium. Thus, the point bears the hallmarks of an unstable fixed point, whilst the parabolas can be considered null clines. However, the language of dynamical systems was in its infancy at this time, and 'unstable fixed point' and 'null clines' were terms Schelling may have been unaware of. The figures certainly have a similarity to the phase portraits of non-linear systems. The arrows suggest the flow of the population and Schelling talks of equilibrium points at (0,50) and (100,0).

Schelling now defines two parameters to explore, namely population size and the tolerance schedule. By levelling the population sizes and increasing the tolerance schedule to a maximum of 5:1 he produces Figure 3.7. An analysis of the this configuration displays what Schelling calls a "mixed equilibrium" [74] at (80,80). Schelling suggests the mixed equilibrium is stable so long as "slightly over 40%" of both colours are present and the entry of one is "not much more rapid" than the other. The arrows would suggest the point corresponds to a stable fixed point.

In another experiment, Schelling explores the effects of limiting the ratios by restricting the entry of a population. So that, for example, if the ratio of white to black is exceeded no more whites may enter (although blacks are free to leave). The result (with symmetric

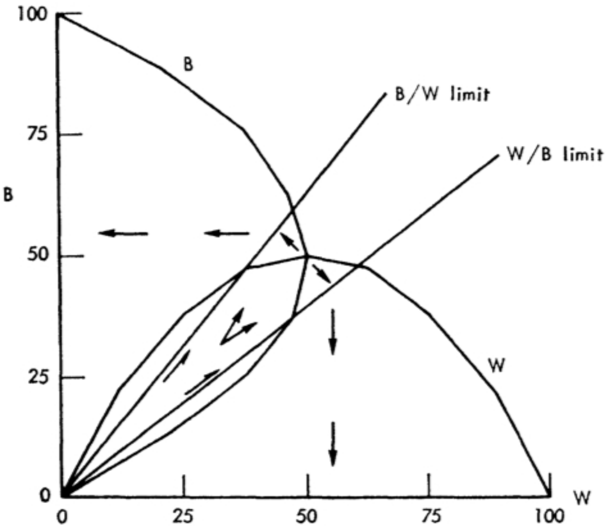


Figure 3.8: Schelling's Bounded Neighbourhood model setup with limits on ratios [74]. When the ratio limit is reached the dominant population can no longer enter, although both populations are free to leave.

populations and tolerance) can be seen in Figure 3.8. The results show that, even though a limit is placed on the dominant population, the final configuration is still complete segregation. This is mainly because, although no more of the dominant population can enter, the ratio is still unacceptable for the minority. This causes them to leave even though no more of the dominant population can enter.

From his experiments, it seems clear that the tolerance schedule has the largest effect on the final system configuration. Although his exploration of the schedule was limited, as he stuck to variants of the straight line tolerance schedule, it offers the opportunity to examine other type of distributions.

3.3 Tipping Model

Schelling's final model [74] can be considered a combination of his two earlier models. By introducing a housing capacity into the environment and altering the rules governing entry of populations, Schelling attempted to recreate the 'tipping phenomenon' which he claimed had been observed in the real world by Mayer [60]. Mayer's study focused on an all white neighbourhood of 700 single family households. The sale of three houses "convinced everyone the neighbourhood was destined to become mixed, within a year 40 households were black and

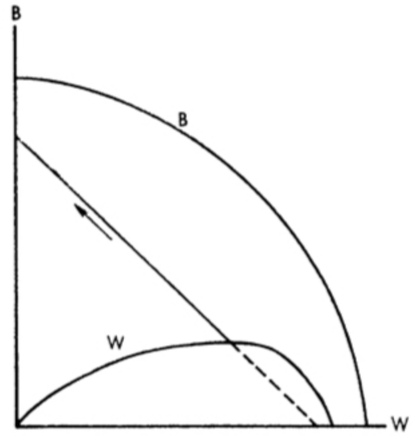


Figure 3.9: Schelling's Tipping Model diagram [74].The line is set at 45 degrees and thus intersects both axes at the same value. The dotted line on the diagram indicates his statically stable mixture whereby both populations are content with the makeup. The arrow again indicates the flow of the population. The lack of an arrow on the dotted line indicates an equal flow in either direction (i.e both populations are equally likely to enter)

within 3 years the number was above 50%" [60]. Schelling claimed the phenomenon is "occasionally observed" [74] in groups such as church groups and is not only ethnically based, pointing out the case of an ice-cream parlour that 'tipped' from a mothers and children clientele to teenagers whose presence was seen as discouraging the mother/children group. Schelling attempts to analyse the phenomenon by using case studies. He notes boundary definitions are more than just that "the neighbourhood has a fixed and well defined [environmental] boundary" [74]. He argues the definition should include individual boundaries, asking "whether the new entrants are clearly recognizable." [74]. Schelling assumes these new entrants are clearly recognisable, which is reasonable enough for populations of whites and blacks. He uses his Bounded Neighbourhood diagrams and implements a limit on numbers in the form of a straight line representing the number of houses (Figure 3.9).

The line is set at 45 degrees and thus intersects both axes at the same value. The dotted line on the diagram indicates his statically stable mixture whereby both populations are content with the makeup. The arrow again indicates the flow of the population. The lack of an arrow on the dotted line indicates an equal flow in either direction (i.e both populations are equally likely to enter). Schelling was only able to produce this result by assuming the minority entering the system had *no preference* towards the neighbourhood makeup (shown by the B curve). This assumption is a two-fold driver towards tipping in that, firstly, according to his entry rules, the demand from the no preference population

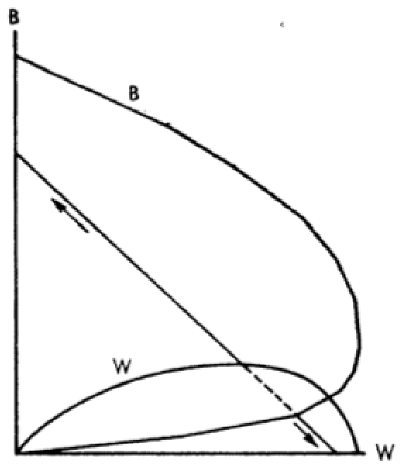


Figure 3.10: Schelling's second Tipping Model diagram [74] with additional preference parameter. Note the final configuration can now be completely W.

is always maximal. Secondly, because of this phenomenon, as soon as one of these agents arrived in the system, the demand from the no preference population will always be stronger than the demand from those who have a preferred makeup.

Schelling then added a preference to both populations. Using the reasonable assumption that the minority preference was not as strong as the majority he produced Figure 3.10. As can be seen the statically stable mixture is reduced and a flow towards an all white population has appeared. This would suggest that even slight preferences in a population are a driver towards segregated communities. However, Schelling's work shows there is no real 'tipping point' as such. The flows are quite smooth and free from any form of jump that would indicate a tipping of a population [74].

3.4 A critique of the Bounded Neighbourhood Model

Schelling's tipping model was the first attempt to apply an ABM to a 'real-world' phenomenon. However the basis of the model was the Bounded Neighbourhood Model. Although the Bounded Neighbourhood Model produces some interesting results, there are a number of problems that must be considered before attempting this kind of application.

1. As shown in §5.1.8, the (100,50) model implies that the difference in the level of tolerance between two adjacent individuals, ranked by tolerance, in the minority community is twice that for two adjacent individuals in the majority community.

2. Not surprisingly a sufficiently high tolerance τ will ensure the existence of subsets of the two communities that can coexist 'stably', in the sense that no-one wishes to leave.

3. The Bounded Neighbourhood Model can be seen as a deterministic model certainly once the movement rule has been clearly defined it is quite simple to work out the results. Whilst this allows stringent testing of the simulation accuracy against the model it is so simplistic that it cannot be said to have any bearing on reality.

4. Although agents distinguish between type, there is no difference within type (*i.e.*, all whites are the same and all blacks are the same).

3.5 Conclusions

The popularity of Schelling's ideas led to criticisms most notably, Yinger [107], who argued that there was no consideration of a number of factors such as economics and social mobility. Similarly, Massey [58], argued that not accounting for environmental/spacial factors and using a homogenous environment meant the model was too simplistic to have any relation to reality. These were the main criticisms, suggesting that simplistic models cannot capture enough information about what really drives segregation. Simple notions of preference, although they probably exist, are superseded by the more important factors of economics and social mobility. Today this could be a fair criticism since the input of experts in the field is an important part of the modelling process. But what Schelling was suggesting with his models is a far cry from the criticisms of Yinger and Massey. Schelling demonstrated that a simple desire for non-minority status in a population could lead to highly segregated systems. However, his attempts to apply his model to reality, by exploring the 'tipping' phenomena of a small case study, was at best clumsy and certainly invited the criticisms. Although Schelling seemed to cover a reasonable exploration of the tipping phenomena, the exploration was limited by his adherence to the straight line tolerance schedule. His failure to explore any other type of distribution could mean that the interesting tipping phenomenon is actually a result of the distribution of the tolerance. Experimentation on the parameters with different distributions might leads to some clarification. Certainly in Schelling's model as soon as an upper bound of tolerance is set, then it becomes a tipping point beyond which no other agent of that type will enter the system. Because of the simplistic nature of his models Schelling was able to isolate and explore abstract ideas of social interaction in Dynamical Models of Segregation [74], a paper that has been cited over two thousand times [79]. Although cautious in his conclusions, this paper was an important chapter in his later book 'Micromotives and Macrobehaviour' [75]. Nevertheless his attempts to apply the model to real world situations by introducing Black and White types and then by applying the model to a single case study, ignited debate and was both championed and derided across disciplines. As we have seen, his critics mainly attacked the simplistic nature of his work, but it is this that allows us to *understand the underlying dynamics explicitly* and so at least produce a primitive framework for models, albeit inadequate, of social behaviour. Schelling acknowledges some of the weakness of the model, highlighting its failure to allow "...for *speculative behaviour*, for *time lags* in behavior, for *organized action*, or for *misperception*"[74, p 181]. It should be noted that Schelling's final conclusion is that the process of tipping is too complex to be treated comprehensively in his paper. The analysis of which:

> "...requires explicit attention to the dynamic relationship between individual behavior and collective results. Even to recognize it when it occurs requires knowing what it would look like in relation to the differential motives or decision rules of individuals."

This is language not out of place in the analysis of complex systems.

3.6 Further work on Schelling

At the time of Schelling's work computers were only able to run simple experiments that could take days to execute and were notoriously fragile. By the mid 90s Epstein and Axelrood at the Santa Fe Institute were using computers able to simulate much larger and more complex systems. Although they never directly worked on Schelling's segregation, their simulations on Sugarscape [31] encouraged others to recreate Schelling's experimentations. While much attention has been given to Schelling's Spatial Proximity Model, there has been less research on what one might consider to be the richer dynamics of the Bounded Neighbourhood Model. There have been a handful of recent attempts to apply some mathematical rigour, most notably Zhang [108] and [109]. Although limited, they offer the beginnings of attempts to give mathematical descriptions to social phenomena and suggest scientifically rigorous approaches in an area that has for too long been unable to do so. The exploration of Schelling's models with new techniques should offer valuable insights into the dynamics of the system. From this a greater understanding of the model and its utility is achieved. Table 3.1 gives a list of authours and the techniques used to model Schelling's segregation. A discussion of the work, and the application of techniques can be found in Chapter 2).

These models are a small, but significant, sample of the current state of research into Schelling's models of segregation. Certainly the work of Stocia and Flache is an interesting

Researcher	Spatial Proximity	Bounded Neighbourhood
Benenson *et al.* [15]	ABM, GIS	-
Laurie and Jaggie [53]	ABM	-
Stoica and Flache [85]	ABM, Game Theory	ABM, Game Theory
Zhang [108]	Game Theory	Game Theory

Table 3.1: Techniques used by researchers developing Schelling's models of segregation.

combination of the two models. However, whilst they all build on Schelling's work, there is little investigation of the Bounded Neighbourhood Model itself.

3.6.1 Stoica and Flache

Stoica and Flache [85] apply Schelling's 'residential' segregation to 'school' segregation. They incorporate Zhang's [108] utility (although here it is a measure of the distance from a school, rather than price), into a combined segregation model. In their description two agent types, A and B, compete over access to 25 schools. Each school has a ratio, initially dependent on the residential ratio. The schools are modelled as points in a Voronoi diagram, with each cell in the diagram considered the related catchment area. A Voronoi diagram is a way of dividing space into a number of regions. A set of points (called seeds, sites, or generators) is specified beforehand and for each seed there will be a corresponding region consisting of all points closer to that seed than to any other. The regions are called Voronoi cells (for more on Voronoi diagrams see [36]). The Voronoi diagram is combined with a Spatial Proximity Model 1000×1000, with each cell considered a residence. The model is then populated with 8700 agents and tested over a number of different experiments.

Initially the utility remains a function of 'ethnic' (A or B) preference in the residential areas. With these setting the model behaves as Schelling's Spatial Proximity Model. This shows the model can be considered a valid Schelling segregation model. Additionally, without the distance preference settings, the schools exhibit Schelling's "tipping process" [85, 4.1].

The author's suggest the results show 'unintended segregation' within the schools. However, more integrated populations arose when agents were allowed into schools outside their catchment area. The work is a simple example of producing a highly stylised 'Schelling' model, which can be used to ask questions about real world school segregation. However, as the authors admit, there are a number of limitations. The tolerance of agents, although varied across experiments, was fixed for the duration of experimentation. Following the Spatial Proximity Model, each agent had an identical tolerance, no ideas of a distribution of tolerance were discussed. In addition the model has a deterministic choice function, which the authors admit, should be replaced with a random utility function akin to Zhang's [108]. The attempts of the authours', to apply the model to 'real world' situations, is hindered by the lack of understanding into some of the workings of Schelling's model. In addition by combining the two models, it is difficult to analyse the effects of space on the Bounded Neighbourhood Model. To gain further insights, it will be important to build a valid simulation of Schelling's Bounded Neighbourhood Model. From this process, some of the mechanics that drive to model should be exposed.

Chapter 4

Methodology

In examining and developing Schelling's Bounded Neighbourhood model it is important to build a solid, defendable framework in which simulations can be developed and run. The simulations can then be used to test hypotheses about the model. Since models are unable to capture all the information within complex systems, there is a need for some level of abstraction. This abstraction will exclude information. In addition, since emergent properties of the system are being sought and so *not explicitly included in the simulation*, it can be difficult to tell if an emergent behaviour is a result of the property being sought, or just created by a bug in the code (or model).[1] Therefore it is important that the development follows a systematic and scientifically sound process. The CoSMoS process has been widely used to model a number of real-world phenomena (see Section 4.3). However, the application of the process to a social model is a novel approach. The CoSMoS process provides a rigorous framework offering a solid foundation necessary for methodological soundness.

4.1 Verification

Verification and validation of computer simulations are important ways of checking for correctness of computer code (in this case a software simulation). Despite thirty years of research in the area, their use is far from widespread. This is possibly because software development in science is typically 'hacked', rather than 'engineered'in a formal sense. While the ideas were not intended for complex systems, they are certainly a useful tool for improving models and their simulations.

Ideas verification and validation of computational accuracy were first defined by Schlesinger,

[1]Whilst this can also be considered an emergent behaviour, we believe it is not the class of emergence we desire.

calling verification the: "...substantiation that a computerized model represents a conceptual model within specified limits of accuracy" [78]. This initial description was simplified by Sargent who argues verification can be defined as: "...ensuring that the computer program of the computerized model and its implementation are correct" [72]. Thus, the verification process is designed to show that the software is internally consistent and correct. This is usually achieved through rigorous testing, using procedures that have been developed and refined in the software engineering field. As with any program there is a need for stringent checks of protocols and structures, to make sure the language is correct; additionally, the program needs to be as bug-free as possible. Tests are done using techniques such as correctness proofs and structured walk-throughs [72]. Rigorous testing of software has developed a number of techniques that can be implemented, including running a trace, entering variables incorrectly, reprogramming critical components and many others [103]. The fact that the verification process is an established knowledge based on years of research means that it can be applied with some confidence.

4.1.1 Prediction

Popper suggests that the power of physics comes from verification through prediction [67]. This is almost impossible with complex systems and certainly not to the degree of accuracy demanded by physics and mathematics. For this reason the issue of prediction is contentious in the field of complexity. Epstein [30] argues that prediction is not the only reason for modelling complex systems and believes that an understanding of the system is much more important. Thompson and Derr [87] counter that with this understanding there must be an ability to predict, citing apparent abilities to predict earthquakes and traffic jams. Troitzsch [88] offers a good summation of the arguments and suggests that without defining what is meant by a prediction the argument is flawed. Using Thompson and Derr's arguments about plate tectonics, Troitzsch offers three different types of prediction:

1. Which kinds of behaviour can be expected [from a system like this] under arbitrarily given parameter combinations and initial conditions?

2. Which kind of behaviour will a given target system (whose parameters and previous states may or may not have been precisely measured) display in the near future?

3. Which state will the target system reach in the near future, again given parameters and previous states which may or may not have been precisely measured?

    (a) Which are the expected value and the confidence interval around the expected

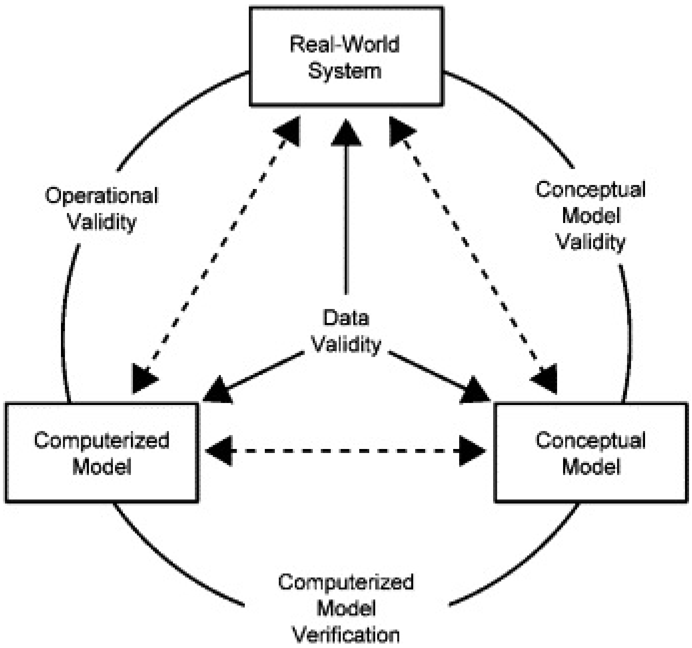


Figure 4.1: Sargent's framework from [72].

value of the state the target system will reach in the near future, again given parameters and previous states which may or may not have been precisely measured?

(b) Which exact value will the state of the target system have at a certain point of time in the near future, again given parameters and previous states which have been precisely measured?

Using the prediction type framework, Troitzch suggests that "the controversy between Epstein and his critics is easily ended. Any kind of good explanation, under all circumstances will yield a prediction of type 1 (and perhaps also of type 2), but not every good explanation will yield a prediction of type 3." [88]

4.2 Validation

The difficulty in the verification of the behaviour of complex systems means that, the validation of models and simulations is even more important. Sargent's framework (Figure 4.1) breaks down validation into three main components. Firstly, conceptual model validation asks not only how well the model represents the held view of reality, but also if the model is any use for the task at hand. Secondly there is internal validation, which determines if the output behaviour can be considered correct. One extreme, but well known, example of internal validation is the Turing Test [89]. This is commonly expressed as the question "Can a human tell the difference between a human response and that of a computer?". More formally, this test utilises experts of a system and attempts to discriminate between the outputs of the system and the model. Finally, there is data validity, which can often be overlooked in circumstances that assume empirical information is 'perfect'. Data validity tries to ensure a number of things: how has the data been collected, has it been collated, normalised or transformed. Are there possible confounds within the data gathering process? Is the variation statistically significant? It should be clear that verification and validation are vital components when simulating complex systems. It is believed that techniques brought from software engineering will help.

The process of verification and validation is inherently difficult when modelling complex systems. The problems are compounded by the absence of a universally accepted framework, so the best course is to adopt a framework that is recognised. The advantages are three-fold. Firstly, since there is no universal framework, a framework that has already been successfully used should be used; since there will be useful information to be gleaned from previous research. Additionally, this should mean a stronger understanding of the suitability for the process. Secondly, by incorporating a personal perspective into the process, the framework can be tailored to the task at hand. Finally, if well built and used, individual frameworks could combine common factors, to produce the emergent framework Epstein [30] hopes for.

4.3 The Complex System Modelling and Simulation process

The Complex System Modelling and Simulation (CoSMoS) process [7] can be seen as a guide for modelling and simulating complex systems, that incorporates verification and validation throughout. The process has already been successfully used in a number of studies including auxin transport canalisation [41], environment orientation [49], immunology [6] and cancer systems biology [19].

The CoSMoS process has three phases that may be nested (from [7, p13]),

Discovery Phase: establishes the scientific basis of the project; identifies the domain of interest, models the domain, and elucidates scientific questions. The phase is concerned with science, not simulation.

Development Phase: produces a Simulation Platform to perform repeated simulation, based on the output of the discovery phase.

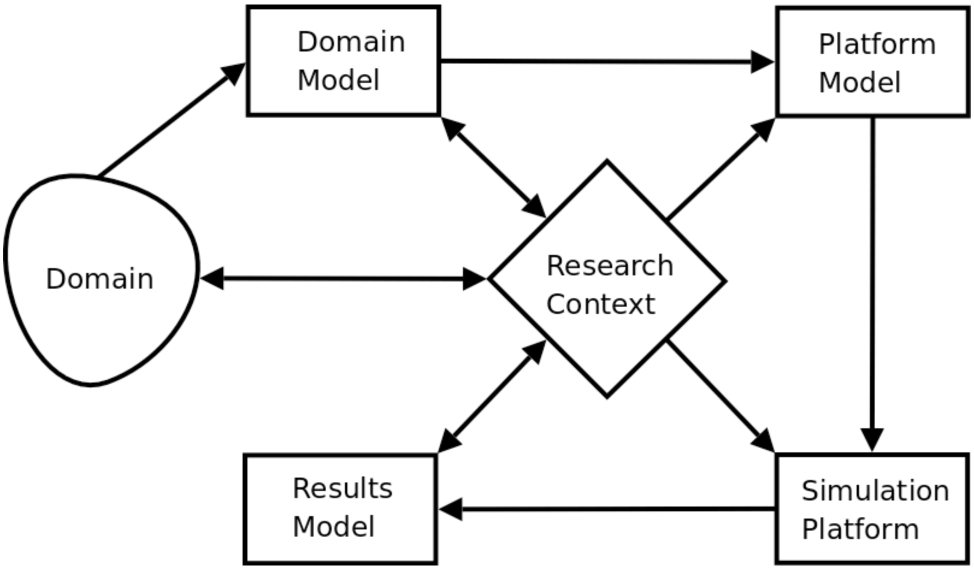


Figure 4.2: The Complex Systems Modelling and Simulation (CoSMoS) process from [7].

Exploration Phase: uses the Simulation Platform resulting from the development phase to explore the scientific questions established during the discovery phase.

In recognition of the challenges involved in understanding and recreating complex behaviours, these phases are not constrained; they can be used iteratively or alternatively, as best suits the problem being considered.

The CoSMoS process identifies products that represent artefacts that are created and modified during these phases. By separating out these artefacts, the process is able to clarify the verification and validation of the model, simulation and results. This separation allows problems to be identified at specific points in the process. The artefacts as defined in [7, p13] are:

The Research Context: captures the overall scientific research context of the project. Including motivation for doing the research, the questions to be addressed by the Simulation Platform, and requirements for validation and evaluation. The research context is discussed in Chapter 2, as well as in the introduction to each experiment (Chapter 6, 7, 8 and 9).

The Domain Model: encapsulates understanding of appropriate aspects of the domain into explicit domain understanding. The Domain Model focuses on the scientific understanding; no simulation implementation details are considered. The Domain Model is described in Chapter 5.1.

Platform Model: comprises design and implementation models for the Simulation Platform, based on the Domain Model and research context. The Platform Model can be UML, pseudo-code or any style that describes the model as it will be run, by the Simulation Platform. This allows the highlighting of possible artefacts arising from the code. The simplicity of Schelling's Bounded Neighbourhood Model means it is easily described using pseudo-code in Chapter 5.2.

Simulation Platform: encodes the Platform Model into a software and hardware platform upon which simulations can be performed. This process is a straightforward process of translating the pseudo-code in Chapter 5.2 into Netlogo [3].

Results Model: encapsulates the understanding that results from simulation: the Simulation Platform behaviour, results of data collection and observations of simulation runs. Note that the way that the Domain Model captures the relevant understanding of the domain is mirrored by the way that the Results Model captures understanding of the Simulation Platform. The results are analysed using the Wilcoxon-Mann-Whitney U test. This nonparametric test is introduced in Chapter 4.4.

The problems discussed by Epstein [30] and Thompson & Derr [87] rather than being systemic, can be addressed at different stages of the modelling process. Thompson and Derr's arguments of verification are based on a simple comparison of their Results Model with actual results. This is problematic on at least a couple of levels. Firstly, if their results differ from expected they will be unable to gauge where the 'fault' lies. Secondly, and more importantly, if their results are as expected, they will still be unable to validate the model or the simulation individually, since it is quite possible an inaccuracy in the model is nullified by an error in the simulation and produces the 'correct' result. This holistic approach to modelling a complex system renders one blind to where anomalies could lie. Without knowing where the problem is, one might have to start again from scratch.

By following the CoSMoS process an important distinction is made, not only between the model and the software that implements the model in the simulation, but also between the results from the simulation and any results from the 'real' world domain. By breaking down the process of modelling, it can be seen that the arguments of Epstein [30], Thompson & Derr [87] and Troitzsch [88] have arisen from an attempt to move straight from the domain to the simulation. Without considering a Domain Model one can see their problems are exacerbated by the attempts to build simulations directly from the domain.

4.3.1 Applying CoSMoS to Schelling's Bounded Neighbourhood Model

As Easterly [28] points out, despite the fact that Schelling's work is so well known, there has been little in the way of empirical evaluation. One reason for this lack of research is due to difficulties with data collection. Although empirical data exist, much of it is in the form of census data. Whilst useful, this data has to be carefully checked and correlated. The process is painfully slow, as great care is needed to make sure transformations (such as extrapolation from sparse data) are applied correctly. There is a risk that alteration of the data can be a subjective process and, as such, introduces bias into the results. More recent data collection and data mining techniques have addressed and reduced the problem by automating many of the processes [102]. However, with human social systems, the timescales involved often means there is only sparse data available. Schelling's models incorporated no external data and he attempted to map his results directly onto actual social systems. It is important to note that it is possible to model systems, even when the understanding of the system is limited. Attempting to build models without following a methodical framework, leads to a weaker understanding of the system, and, in many ways, nullifies any conclusions. There is a risk that failure to adhere to good practice will lead to a plethora of invalid, and unverified, models. Results from these experiments could tarnish the reputation of the field. The Complex Systems Modelling and Simulation (CoSMoS) process is established and recognised within complexity community as a reputable process. Building a rigorous model using the CoSMoS process enhances our understanding, as well as offering protection against weakness resulting from poor validation and verification.

4.4 Hypothesis Testing

The analysis of the results is key evidence for any experiment. If there is to be a comparison between the different model variations, the results must be analysed in a consistent manner. The experiments presented in Chapters 6, 8, 7 and 9 are analysed using hypothesis testing. In each case we present two results, static and dynamic. The results are used to provide evidence towards refuting the null hypothesis (H_0), that any changes in the results are not significant.

4.4.1 Static result

The static result measures the final population configuration, M, of two populations, Orange (O) and Blue (B), in an environment E (presented in detail in Chapter 5). Following work

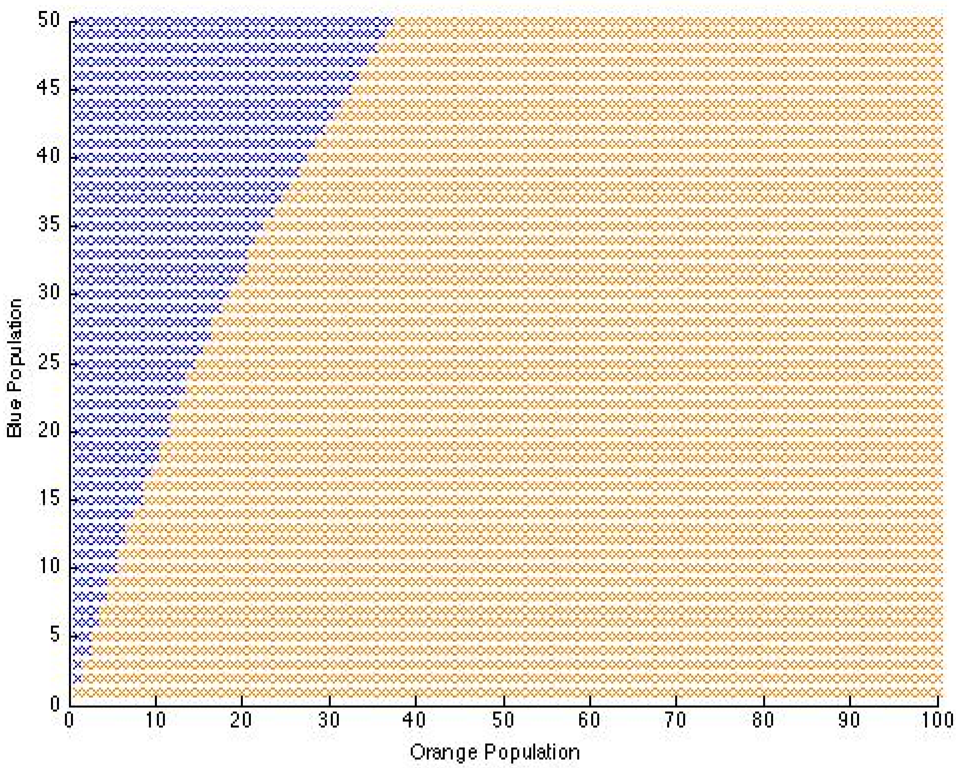


Figure 4.3: Static results of Schelling's Bounded Neighbourhood Model. The initial conditions of the model are marked by the colour of the final population from that state. A clear boundary can been seen along the $1.5O : 1B$ ratio suggesting the B population have to be a clear majority if they are to survive.

by Freeman [38],

$$M = \frac{2(|(O \cap E)| * |(B \cap E)|)}{|(O \cup B)|(|(O \cup B)| - 1)}, \tag{4.1}$$

where $O \cap E$ and $B \cap E$ are the two populations being measured. Values of $M > 0$ indicate the system produces mixed populations. Taking $O = 100$ and $B = 50$, and using Schelling's movement rules (move in order of tolerance), the model runs through to a final population configuration. By plotting the colour of the final population on the initial condition location, a colour map is developed (see Figure 4.3). The calculation of M yields 0, meaning only segregated populations are produced, correlating with Schelling's results.

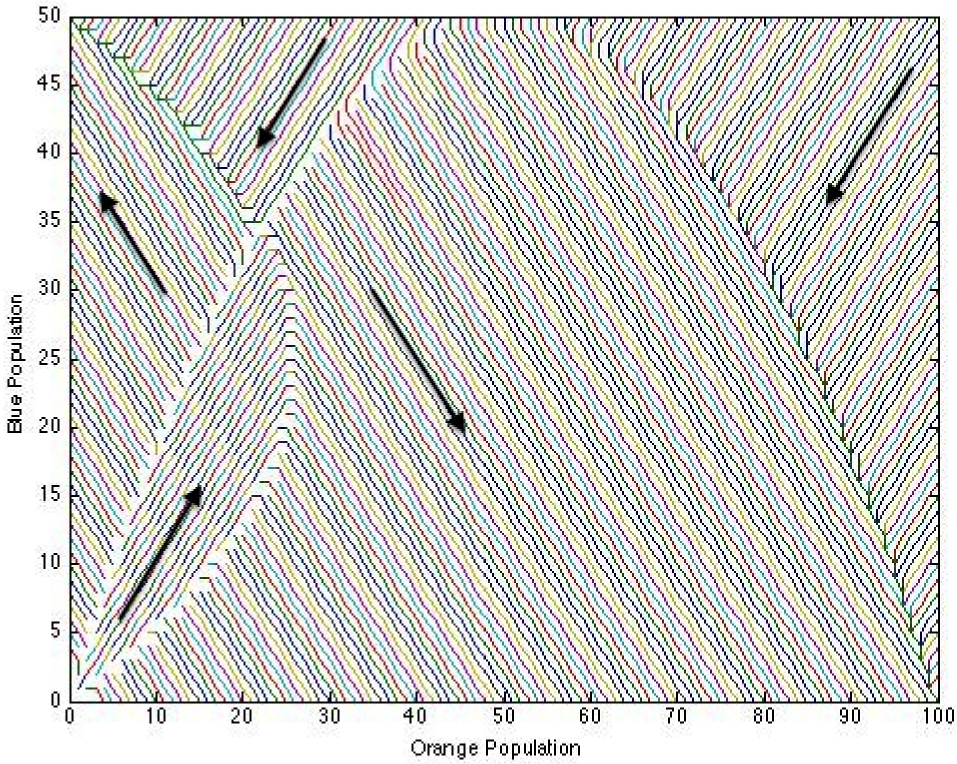


Figure 4.4: Dynamic results of Schelling's Bounded Neighbourhood Model. From initial conditions of the model, population size is measured and plotted at each iteration. The results recreate Schelling's boundaries and flows.

4.4.2 Dynamic result

The dynamic result analyses the flows of the populations into (and out of) the neighbourhood. The statistic I is a measure of the number of iterations taken for the system to reach a steady state. A steady state is reached when no more agents wish to move. The results from the dynamic measure can be presented in one of two ways. For example, plotting the path of the populations on a two dimensional plane, from all possible initial conditions, recreates Schelling's flows (see Figure 4.4). A second option is a cumulative distribution of I. This is a useful visualisation tool, as one can quickly see any changes in the results. Testing the result means calculating the distribution of runtimes. Calculating all the initial conditions possible with $100O, 50B$, gives 5000 results, these can be plotted to produce a cumulative frequency distribution (see Figure 4.5). Since Schelling's model is deterministic, there is no

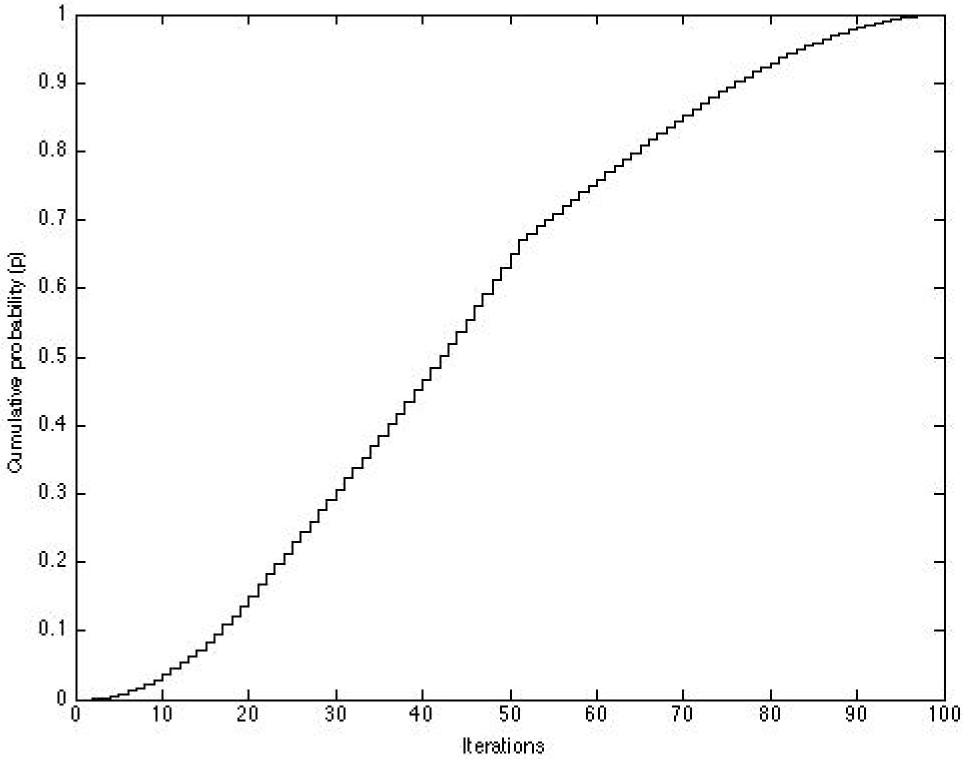


Figure 4.5: Plotting the results as a cumulative frequency distribution allows easy visualisation around the spread of results from different initial conditions.

distribution around runtimes of individual initial conditions. The introduction of stochastic processes, creates the need for repetition of runs; each initial condition is repeated 25 times. These repetitions guard against chance results, and strengthen the evidence provided. The hypothesis, that changes to the model change I, is tested using the Wilcoxon-Mann-Whitney U test, which will be referred to as the U test.

4.4.3 The Wilcoxon-Mann-Whitney U test

The U test is a nonparametric test used to analyse the result of the simulation. Unlike parametric tests, the U test allows analysis of populations when their distributions are unknown. In addition, because the analysis uses the medians, the test is more robust against reporting significant results due to outliers. A good discussion of parametric and nonparametric tests can be found in [8]. In this case the advantages over parametric tests are:

score	8	9	10	11	12	13	14	15	16	17	18	19
group	Y	X	Y	X	Y	Y	X	X	X	Y	Y	X
rank	1	2	3	4	5	6	7	8	9	10	11	12

Table 4.1: Results in ranked order

- The U test uses the median of samples rather than the mean, thus it is more robust against outliers. However the U test does assume that the samples are independent.
- There is no assumption of any particular underlying distribution. Rather, the two samples being compared are assumed to have the same distribution.
- When analysing large samples non-parametric tests are just as powerful as parametric tests.

For example, suppose we draw two lots of sample data,

$$X = 9, 14, 15, 11, 16, 19$$

$$Y = 8, 10, 12, 13, 17, 18.$$

The null hypothesis (H_0) is that the two samples X and Y are drawn from the same population. The alternative hypothesis (H_1) is that X is stochastically larger than Y, (*i.e.*, $P[X > Y] > 0.5$). To test the hypothesis the U statistic is calculated, this is done by arranging the results in ranked order (table 4.1).

For each observation in X, we count the number of data points in Y that have a smaller rank, giving $U_X = 1 + 2 + 4 + 4 + 4 + 6 = 21$. Similarly, for each in Y we count those in X with a smaller rank, giving $U_Y = 0+1+2+2+5+5 = 15$. Consulting the significant tables with $\#X = \#Y = 6$ and a 95% confidence level, shows that a significant results requires $5 \leq U_Y \leq 32$. Since $U_Y = 15$, we are unable to refute the null hypothesis that the samples are drawn from populations with the same medians.

4.4.4 Increasing sample size

One problem with statistical analysis arises when the number of data points increases to a large number. For example, consider the case $\#X = \#Y = 5000$ (Figure 4.6. The results of the U test give a significant result, refuting the null hypothesis. However this could easily be an artefact of the number of data points. Indeed a comparison of the two distributions

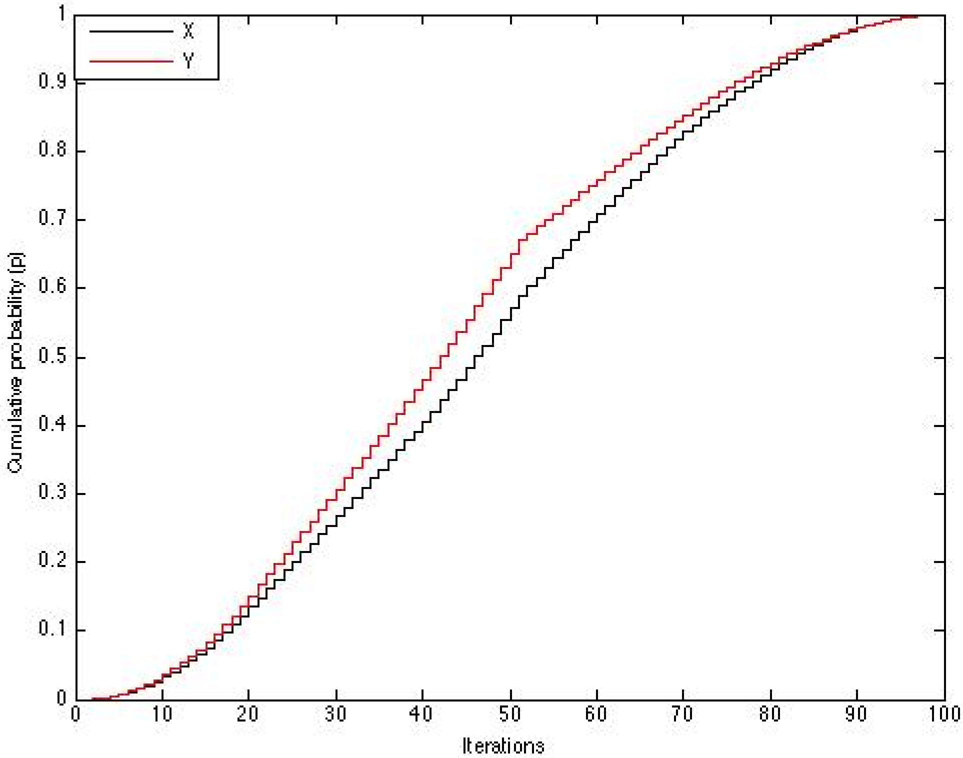


Figure 4.6: A comparison of the distributions of X and Y. Results of the U test show a statistically significant difference ($p < 0.05$), but a small effect size ($A = 0.54$). This can be seen in the similar distributions.

(Figure 4.6) actually shows similar distributions. To check the effect size a second statistic is needed.

4.4.5 Vargha and Delaney A statistic

The Vargha and Delaney A statistic quantifies the effect size in comparing statistical data. The effect size is based on the *measure of stochastic superiority* equation [91, eq (2)]:

$$A_{12} = P(X > Y) + .5P(X = Y),$$

where X and Y are samples from populations 1 and 2 respectively. It can be seen that when the two populations are stochastically equivalent, $P(X > Y) = (PX < Y)$. By calculating the difference in these probabilities, Vargha and Delany classify three effect sizes:

- Small - $A_{12} > 0.56$ or < 0.44,
- Medium - $A_{12} > 0.64$ or < 0.36,
- Large - $A_{12} > 0.71$ or < 0.29.

Applying the A statistic to the results shown in Figure 4.6 returns a value of $A_{12} = 0.54$, which indicates there is less than a small effect size. This provides evidence that although we reject the null hypothesis due to a difference in the results, the difference could just be an error produced by the amount of data. Now consider X and a new data set Z; again the U test returns a significant result. A comparison of the distribution indicates a difference. Applying the A statistic returns a value of $A_{12} = 0.75$ indicating a large effect size. In this case the null hypothesis is rejected and the A statistic provides evidence that the result is interesting. A comparison of the distributions (Figure 4.7) shows a clear difference. The U test and the A statistic are used in the analysis of I later in this thesis.

4.5 Experimental structure

The work presented here explores some of the assumptions in Schelling's Bounded Neighbourhood Model, through a set of experiments. The work follows a cyclic path, as suggested by the CoSMoS process. Initially, in a novel use of the CoSMoS process, Schelling's social model is made explicit and used as the Domain model, presented in Section 5.1. The formalisation is translated from Schelling's original paper [74], and its development highlights some assumptions in the model. Schelling makes four basic assumptions in his paper. A chapter is devoted to each of these assumptions, directly quoted from Schelling's paper, which are subjected to computational experiments. Changes are made to the parameters and structure of the domain model, derived from Schelling's model, to enable a thorough testing of the assumptions. Following the development of the Domain Model (and in line with the CoSMoS process), a Platform Model is developed in Chapter 5.2, the Simulation Platform is described in Chapter 5.3 and the Results Model in Chapter 5.4. The analysis of the results will be undertaken using the U test and, where necessary, A statistic; the population flows and final population composition are analysed in an identical manner. The statistical tests, used to analyse the results, are described in Section 4.4, with ideas from the results of the tests compared back to Schelling's model and used to inform the direction of the next experiment. At the end of each cycle, the results feed back into further experimentation and another iteration of the process begins.

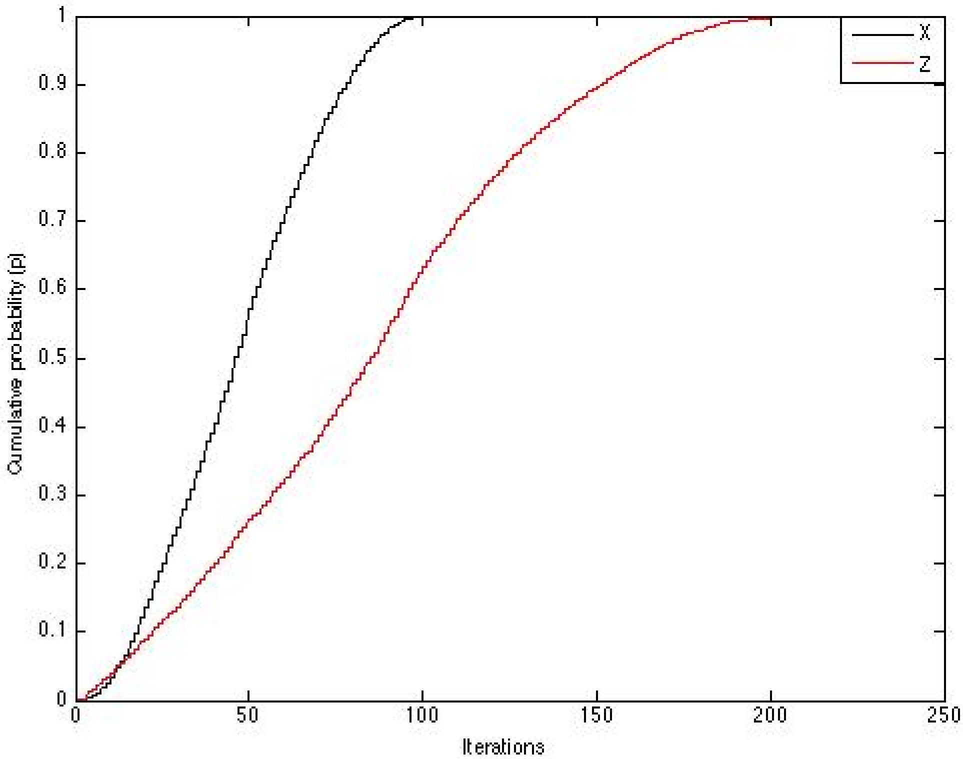


Figure 4.7: A comparison of the distributions of X and Z. The clear differences in the distributions is shown by the U test returning a statistically significant difference ($p < 0.05$), but now there is a large effect size ($A = 0.75$).

Chapter 5

Formalising the Bounded Neighbourhood Model

Schelling's Bounded Neighbourhood model [74, p167], discussed in Chapter 3.2, considers heterogeneous population flows into and out of a neighbourhood or *environment* E (defined in Chapter 5.1). It is aspatial, in that agents consider the composition of agents in E and not of the total population.

Each agent calculates their 'happiness' (defined in Chapter 5.1.9), which is based on the current ratio of the number of agents like themselves to those unlike themselves inside E. This ratio is then measured against their individual tolerance of such a ratio, thus determining their 'happiness'. Sad agents inside E leave, while 'happy' agents outside E enter.

5.1 Developing the Domain Model

Schelling's Bounded Neighbourhood model is used as the basis of a formal Domain Model, bringing it up to the level of precision required for the CoSMoS process. Applying the process to Schelling's model (Figure 5.2) reveals the shortcomings in his model, theoretical simulation and results.

Firstly, and most importantly, Schelling's first step from the domain (Segregation) to the Domain Model (informal Bounded Neighbourhood Model) is Schelling's personal interpretation [74, p143]. Whilst this is acceptable (but not advised) for models in which the developer is a domain expert, Schelling's background was mathematical, rather than sociological. However Schelling's model was developed at a time when cross-disciplinary approaches were rare and, as such, this was a failure of the system rather than the authour. Secondly, Schelling's 'formalisation' contains ambiguities that, if carried through to the simulation, could produce

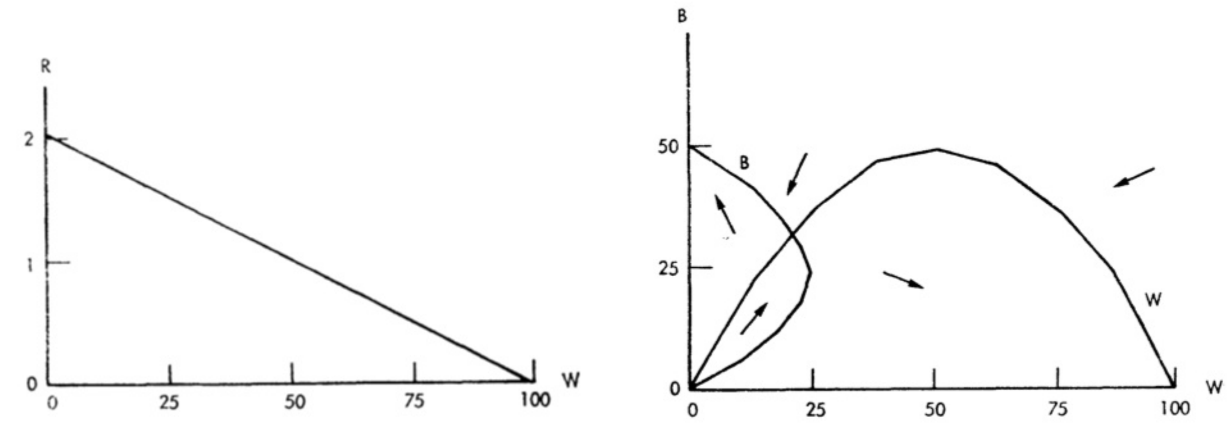


Figure 5.1: Schelling's Bounded Neighbourhood Model: Tolerance schedule (left) and its realisation (right) [74, fig. 18].

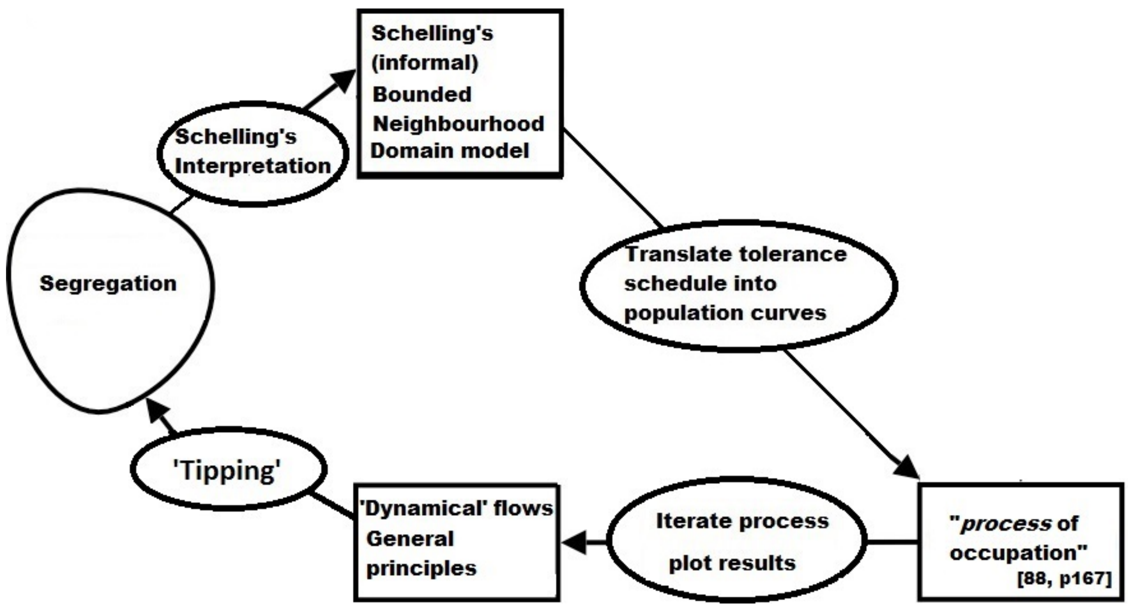


Figure 5.2: A diagram of Schelling's process in developing the Bounded Neighbourhood Model.

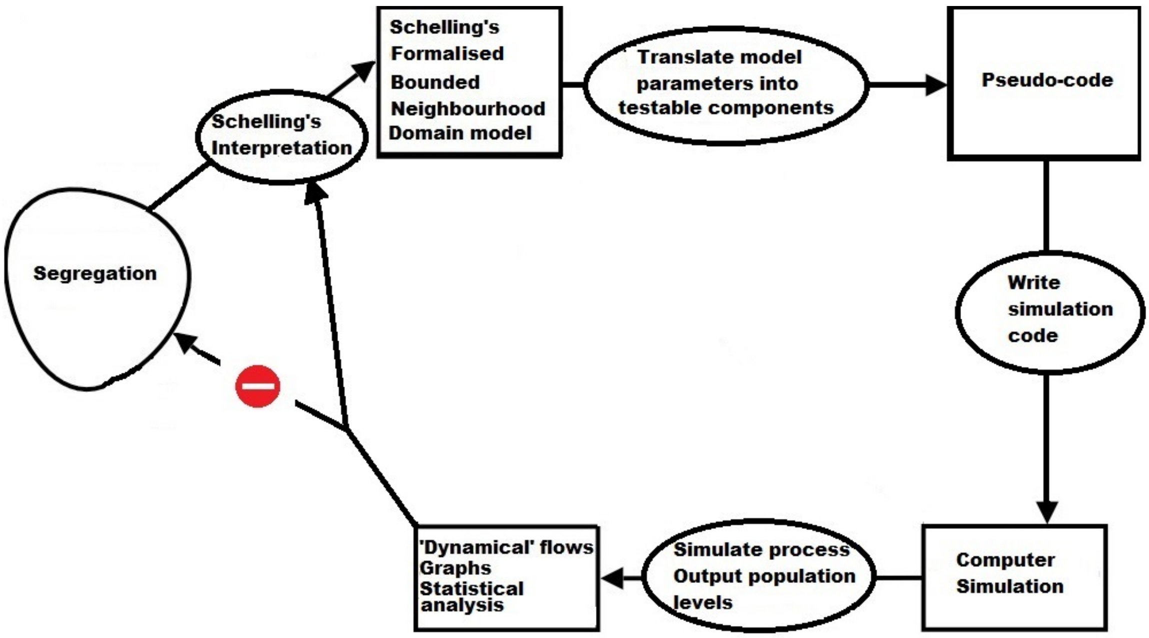


Figure 5.3: An application of the CoSMoS process in developing a simulation of Schelling's Bounded Neighbourhood Model. Note the outputs of the process feed back into the interpretation, rather than the real world.

differing results. Finally, it is clear that Schelling takes his results and then attempts to apply them to a real world event (neighbourhood tipping) [74, p181]. Because of his failure to collaborate in this first instance, it is difficult to defend attempts to relate his results back to reality. This argument is used by a number of authours, who attack his models as missing vital components of reality (see Chapter 3 for further discussion).

In order to strengthen the model we adapt Schelling's process so that it fits with the CoSMoS process (Figure 5.3). However, whilst this adaption is an attempt to improve Schelling's model, his interpretation is retained. *Because of this, and the inherent difficulty of relating models to reality, we make no attempt to relate the model back to reality.* Instead, results from the simulation are used to formulate questions about Schelling's model. It's important to stress that, although at times wording is used that may be interpreted as anthropomorphic (e.g., Blacks and Whites or 'happy or unhappy' etc.) there is no claim that agents are mirroring individuals. However, the fact that this needs to be highlighted indicates the ease in which the lines between the model and reality can be blurred. The application of the CoSMoS process should guard against this. The first step in this process is developing a formalised Domain Model, the basis of this model will be Schelling's own parameters.

Neighbourhood. Schelling describes this as a particular bounded area that is preferred to any alternative (where possible in this thesis we avoid terminology with spatial connotations). He gives as examples membership of a club or participation in a job.
"A person is either inside it or outside it" [74, p167].

Colour Ratio. It is measured against agents' tolerance to calculate the state of the agent.
"Absolute numbers do not matter, only ratios." [74, p167]

Tolerance. A limit that, once reached, triggers agent movement.
"He will reside in it unless the percentage of residents of opposite color exceeds some limit. Each person, black or white, has his own limit. ('Tolerance,' we shall occasionally call it.) If a person's limit is exceeded in this area he will go someplace else" [74, p167].

Process of occupation. Population movement into and out of the system. This is the process Schelling was most interested in,
"What we can do is look at the *process* by which the area becomes occupied" [74, p167].

Following these, Benenson [15] suggests some "standard settings" for a Schelling segregation model. Although based on Schelling's Spatial Proximity Model (see Chapter 3.1), the settings (set out below) are used as the basis for the formalisation of the Bounded Neighbourhood Model. The major differences between the models are the agents' consideration/definition of neighbourhood and their tolerance. Although both use the same concept of Neighbourhood Ratio to determine 'happiness' (defined below), the models explore different dynamics. The Spatial Proximity Model looks at the dynamics *within* a neighbourhood, whereas the Bounded Neighbourhood Model explores the effects of agents' movement *into and out of* a neighbourhood. Agent 'happiness' is measured by comparing the tolerance of the agent with the current ratio of the neighbourhood. If the neighbourhood ratio is above the level of the agent's tolerance then the agent is deemed 'unhappy' and will attempt to move.

The Bounded Neighbourhood Model parameter settings are described below:

The Environment. The environment is defined as a global neighbourhood, in which agents are either inside or outside.

Neighbourhood Ratio. This is the number of agents of other type compared to the number of own type inside the neighbourhood.

Agent Location. Agents are either inside or outside of the neighbourhood.

Agent Tolerance. Agents' level of tolerance of the ratio of their neighbourhood. Unlike the Spatial Proximity Model, where agents all have the same tolerance, here the agents' tolerance is defined as a cumulative straight line schedule (Figure 5.1). Individual agents *of the same type* have a different level of tolerance. Two agents of different type could have the same level of tolerance.

Agent Dynamics. The 'unhappy' Bounded Neighbourhood Model agents (within the neighbourhood) remove themselves one by one, starting with the most 'unhappy' of the population. If there are agents outside the system who would be 'happy' to be inside, they move in one at a time starting with the most 'happy'.

By using an aspatial model, Schelling is able to extend the model beyond spatial grids. This allows the encapsulation of groups that are not necessarily spatially linked. The model can be described by the following characteristics.

5.1.1 The Environment

The model has two aspatial regions (sets of agents), the neighbourhood E (agents 'inside') and its complement E^* (agents 'outside').

5.1.2 Agent Location

Agents are located in E or in E^*. Agents' decisions are based solely on the composition of E.

5.1.3 Agent Type

Schelling's model explores ideas of segregation with two distinct populations distributed inside the neighbourhoods E. The populations consist of two types or sets W and B (it is natural to think that B represents the set of 'blacks' and W the set of 'whites'):

$$\mathrm{TYPE} = \{B, W\}. \tag{5.1}$$

Clearly this can be generalised to any number of types:

$$\mathrm{TYPE} = \{T_1, T_2, \ldots, T_n\}\ ; \qquad n \geq 2. \tag{5.2}$$

5.1.4 Agent numbers

The set of all agents is $B \cup W = E \cup E^*$ (all agents are either inside or outside); their number is $|B| + |W|$. The number of agents in B and in E (the number of black agents inside) is $|(B \cap E)|$, the number of agents in W and in E (the number of white agents inside) is $|(W \cap E)|$. In his initial experiment Schelling [74, fig.18] chooses $|B| = 50$, $|W| = 100$.

5.1.5 Ratio of the two populations

Schelling defines the Ratio R_{WB} for B to be the number of white agents inside compared to the number of black agents inside. This can be formalised as

$$R_{WB} = \frac{|(W \cap E)|}{|(B \cap E)|} \tag{5.3}$$

and the ratio R_{BW} for W is the reciprocal of R_{WB},

$$R_{BW} = \frac{|(B \cap E)|}{|(W \cap E)|}, \tag{5.4}$$

provided that the denominators do not vanish. The case when they do vanish is discussed in Chapter 5.1.6.

In the general case, R_{ij} is the ratio of the number of all agents of type j to the number of type i:

$$R_{ij} = \frac{1}{|(T_i \cap E)|} \sum_{j \neq i} |(T_j \cap E)|, \tag{5.5}$$

provided $T_i \cap E \neq \emptyset$. Here we also assume that all other types are equally weighted (equally disliked); a more complicated model could include a weighting factor $z_{ij} (i \neq j)$ for each type relative to each other type.

5.1.6 The tolerance schedule and definition of τ

Schelling assigns each agent a tolerance, which we denote by τ. The tolerance of an agent w is thus $w(\tau)$ and that of an agent b is $b(\tau)$. These tolerances are defined by a schedule comprising a straight line cumulative frequency distribution[1] (see equations (5.6) and (5.7) below). Here we quote the relevant passage from [74, p.168], with our parenthetical additions of specific model parameters and values:

> We can take the total of whites to be 100 [$|W| = 100$]. Suppose that the median white [w_{50}] is willing to live with blacks in equal numbers [has $\tau = 1$], so that 50 among our 100 whites [w_1 to w_{50}] will abide a ratio of black to white of 1.0 or greater. The most tolerant white [w_1] can abide a black-white ratio of two to one

[1]Strictly speaking a cumulative distribution function must be monotonic increasing; and would be if the axis was "intolerance".

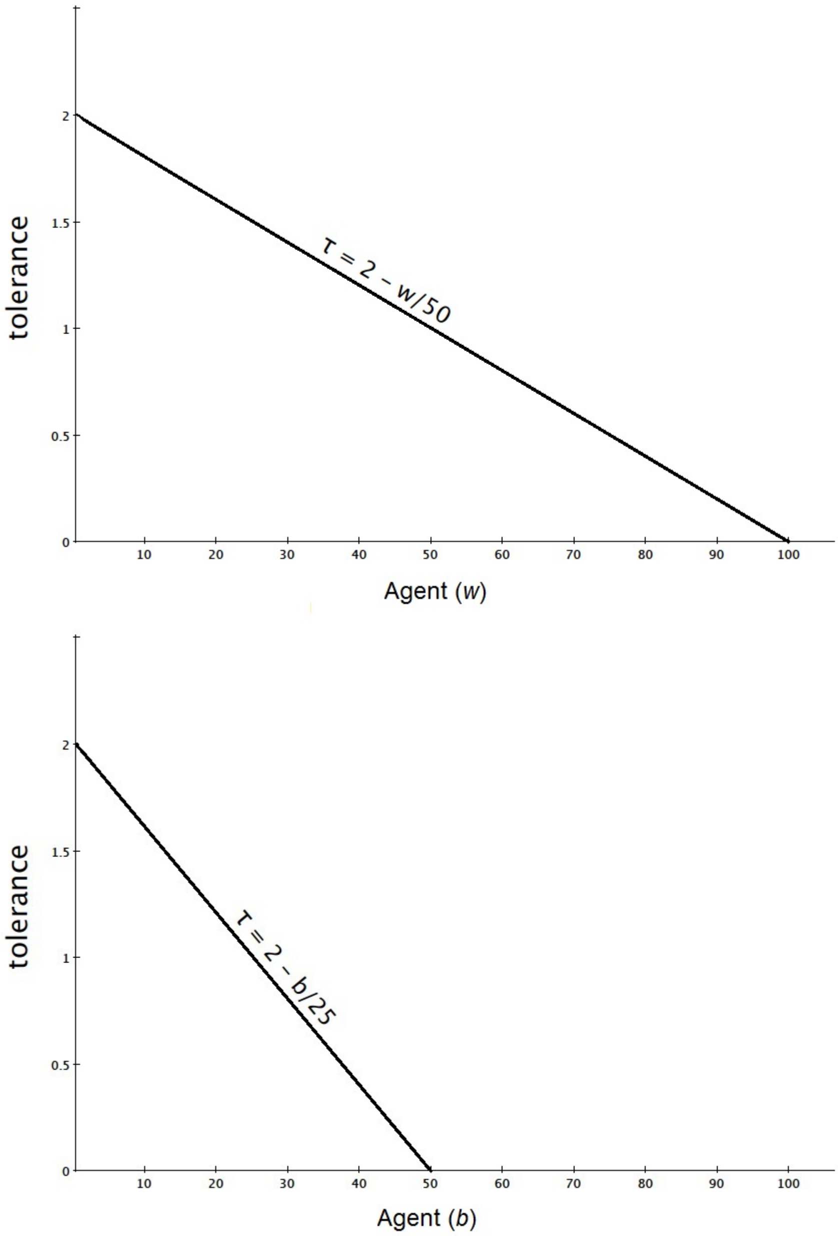


Figure 5.4: The straight line tolerance schedules of the W (top) and B (bottom) populations. The horizontal axis represents agents w_1 to w_{100} (top) and b_1 to b_{100} (bottom). The pronounced difference in the angle of the slopes is clear.

> [has $\tau = 2$], that is, is willing to be in a one-third minority; and the least tolerant white [w_{100}] cannot stand the presence of any blacks [has $\tau = 0$]. The cumulative distribution of tolerances for a white population will then appear as in the top of Figure 18 [our Figure 5.1]. It is a straight line with intercept at a ratio of 2.0 on the vertical axis and intercept on the horizontal axis at the 100 whites who comprise the white population.

In other words, the W population is ordered according to tolerance, and giving them a numerical identification according to its position in the line. The first (most tolerant) agent (w_1) has $\tau = \tau_{max} = 2$, the last (least tolerant) agent (w_{100}) has $\tau = 0$, and the intermediate agents have τ values defined by a linear interpolation. This is shown in Figure 5.4. The horizontal axis represents the numerical identification $w_1 - w_{100}$ of agents, in order of tolerance. The vertical axis τ represents the maximum ratio an agent will tolerate. The mathematical formulation gives the linear equation

$$w_n(\tau) = 2 - \frac{w_n}{50}, \tag{5.6}$$

so that, for instance, w_{100} has tolerance $\tau = 0$ (Figure 5.4). However there is a problem concerning the origin. It is clear from Schelling's sentence

> "The most tolerant white ... is willing to be in a one-third minority;" [74, p.168],

he assumes that w_1 has a tolerance $\tau = 2$ in contradiction to (5.6), where when $w_0(> w_1)$ has tolerance $\tau = 2$. The formula can be reconciled with the above sentence either by altering the end points or the slope. For instance the linear schedule can be chosen to pass through the points $(1, 2)$ and $(100, 0)$ in which case the slope is $-2/99$ and the tolerance τ is given by

$$w_n(\tau) = -\frac{2w_n}{99} + \frac{200}{99}.$$

Nevertheless we will continue to use equation (5.6) for ease of comparison with Schelling's paper (noting that the relatively small discrepancy does not affect the conclusions drawn).

Schelling's initial model also uses a similar tolerance schedule with the same maximal tolerance $\tau = 2$ but with $|B = 50|$ and with slope $-1/25$. This gives the lower of the two diagram shown in Figure 5.4. The mathematical formulation gives the linear equation

$$b_n(\tau) = 2 - \frac{b_n}{25}, \tag{5.7}$$

so that, for instance, agent b_{50} has $\tau = 0$ (again, the same problems with the origin, as mentioned earlier, arise and are ignored).

Schelling also explores variations of the tolerance schedule, including $\tau_{max} = 5$ with $|W| = |B|$ [74, fig.19], and other values of τ_{max} and other total population ratios IWI/IBI. He also discusses a non-linear (but still monotonic) tolerance schedule [74, fig.25]. Stably integrated populations can be obtained by simply increasing the tolerance of agents to extreme levels. In this thesis, Schelling's original numbers and ratios are addressed in an attempt to produce integrated populations without significant changes in the tolerance (or numbers) of agents.

5.1.7 Translation of the tolerance schedule into population numbers

We follow Schelling in the translation of the linear tolerance schedule (5.6). By his definition of tolerance a white agent w_n has a tolerance τ

$$w_n(\tau) = 2 - \frac{w_n}{50}. \tag{5.8}$$

However it is also the case that the tolerance of an agent w_n can also be calculated as

$$w_n(\tau) = \frac{b_n}{w_n}, \tag{5.9}$$

where b_n is the number of corresponding blacks. This implies

$$w_n(\tau) = 2 - \frac{w_n}{50} = \frac{b_n}{w_n}. \tag{5.10}$$

Hence solving (5.10) we get the parabola

$$b_n = w_n \left(2 - \frac{w_n}{50}\right), \tag{5.11}$$

which passes through the origin $(w_n, b_n) = (0, 0)$ and the point $(100, 0)$, with a maximum at the point $(50, 50)$ (Figure 5.1).

Similarly, if we consider the tolerance $b_n(\tau)$, the tolerance schedule implies that the tolerance at b_n is given by

$$b_n(\tau) = 2 - \frac{b_n}{25} = \frac{w_n}{b_n}, \tag{5.12}$$

by Schelling's definition of tolerance, where w_n is the number of corresponding whites. Hence solving (5.12) we get the parabola

$$w_n = b_n \left(2 - \frac{b_n}{25}\right) \tag{5.13}$$

which passes through the origin $(w_n, b_n) = (0, 0)$ and the point $(0, 50)$, with a maximum at the point $(25, 25)$ (Figure 5.1).

5.1.8 Agent Tolerance

According to Schelling [74, p.168]

> [An individual] will reside in [the Neighbourhood] unless the percentage of residents of opposite colour exceeds some limit. Each person, black or white, has [their] own limit. ('Tolerance,' we shall occasionally call it.)

Thus Schelling defines individual τ. To make Schelling's notion of tolerance clearer, consider the tolerance $w_n(\tau)$ of the individual w_n. The next individual at w_{n+1} will have tolerance $w_n(\tau - 1/50)$ (Figure 5.4). For the B population, $b_{n+1}(\tau) = b_n(\tau - 1/25)$ (Figure 5.4). We assume that each agent has a single tolerance τ_i for other agent types T_j $(i \neq j)$; a more complicated model could include a different tolerance for each type.

5.1.9 Agent Happiness

The 'happiness' of an agent w_n is dependent on the Neighbourhood Ratio R_{BW} and on the agent's tolerance $w_n(\tau)$ for that ratio. An agent is *'happy'* precisely when $R_{BW} \leq w_n(\tau)$, *i.e.*,

$$w_n(happy) \iff R_{BW} \leq w_n(\tau). \tag{5.14}$$

Similarly, the 'happiness' of an agent b_n

$$b_n(happy) \iff R_{WB} \leq b_n(\tau). \tag{5.15}$$

Note that the 'happiness' depends only on the ratio *inside* the neighbourhood; what happens outside the neighbourhood does not effect agent 'happiness'. Also, the model does not allow agents to specify a minimum ratio: "there are no minority-seeking individuals" [74, p.167]. Therefore the model is already biased towards segregation.

5.1.10 Agent Dynamics

The parameters above give the properties of the agent populations; the movement rules define the dynamics, *i.e.*, define how the agents move into and out of the neighbourhood (what Schelling calls a "process of occupation" [74, p170]). The movement is decided by the agents' location and 'happiness' (itself a function of the agent's tolerance and the relevant Neighbourhood Ratio). Agents can either enter the neighbourhood, stay where they are, or leave the neighbourhood. If an agent is inside the neighbourhood and 'unhappy' it can leave; those outside who would be 'happy' to be inside will enter. Agents who are inside and 'happy' will stay, and those outside who would be 'unhappy' inside will remain outside. The agent movement rules are described by Schelling [74, p.170] (again, with our parenthetical additions of specific model parameters) as:

> if all whites present in the area are content [$\nexists w_n \in (W \cap E)$ with $w_n(\tau) < R_{WB}$], and some outside would be content if they were inside [$\exists w_n \in (W \cap E^*)$ with $w_n(\tau) \geq R_{WB}$], the former will stay and the latter will enter in order of tolerance [highest τ first]; and whites will continue to enter as long as all those present are content and some outside would be content if present. If not all whites present are content [$\exists w_n \in (W \cap E)$ with $w_n(\tau) < R_{WB}$], some will leave; they will leave in order of their discontent [lowest τ first], so that those remaining are the most tolerant; and when their number in relation to the number of blacks is such that the whites remaining are all content [$\nexists w_n \in (W \cap E)$ with $w_n(\tau) < R_{WB}$], no more of them leave. A similar rule governs entry and departure of blacks [agents of type B].

This needs further refinement to make it a properly defined algorithm. We assume from the phrase "they will leave in order of their discontent" that one agent enters or leaves per iteration, and that R is recalculated each iteration. The questions of the handling of different agent types and the movement of agents after each iteration, are unexplained[2].

The movement of a single agent of a given type is illustrated in Figures 5.5 and 5.6. Here we assume that all the agents inside have a higher tolerance than all the agents outside. Then it cannot be the case that there are simultaneously 'unhappy' agents inside and 'happy' agents outside: either (a) all inside are 'happy', or (b) all outside are 'unhappy'. Once achieved, the movement rules maintain the order. Having developed the formalisation the next step is to develop the Platform Model and simulation to test the effect of the movement rule.

5.2 Platform Model

With the Domain Model formalised, and the parameters identified, we can now develop a Platform Model. The Platform Model is an abstract representation of the dynamics of the parameters, and their interactions. This representation is abstract enough to be applicable to

[2]Testing these questions in the simulation shows no change in final outcomes. However, as expected I is larger for single movement compared to interleaved. The difference in the population flows suggests Schelling's model used interleaved movement.

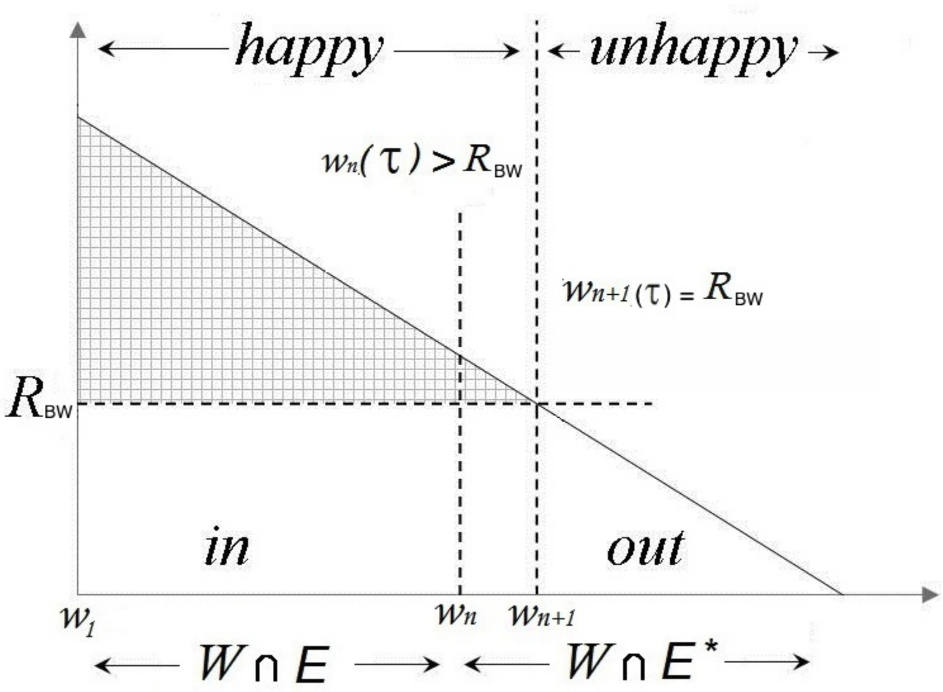


Figure 5.5: Movement rules for the W population derived from Schelling's description (see Chapter 5.1.10). 'Happy' agents lie below the hatched area. The ratio R_{BW} causes one agent (w_{n+1}) outside to be 'happy', so w_{n+1} enters the neighbourhood E. This increases $W \cap E$ by one, thereby decreasing R_{BW} so potentially making more W agents 'happy', and so on. The B population behaves in an identical manner.

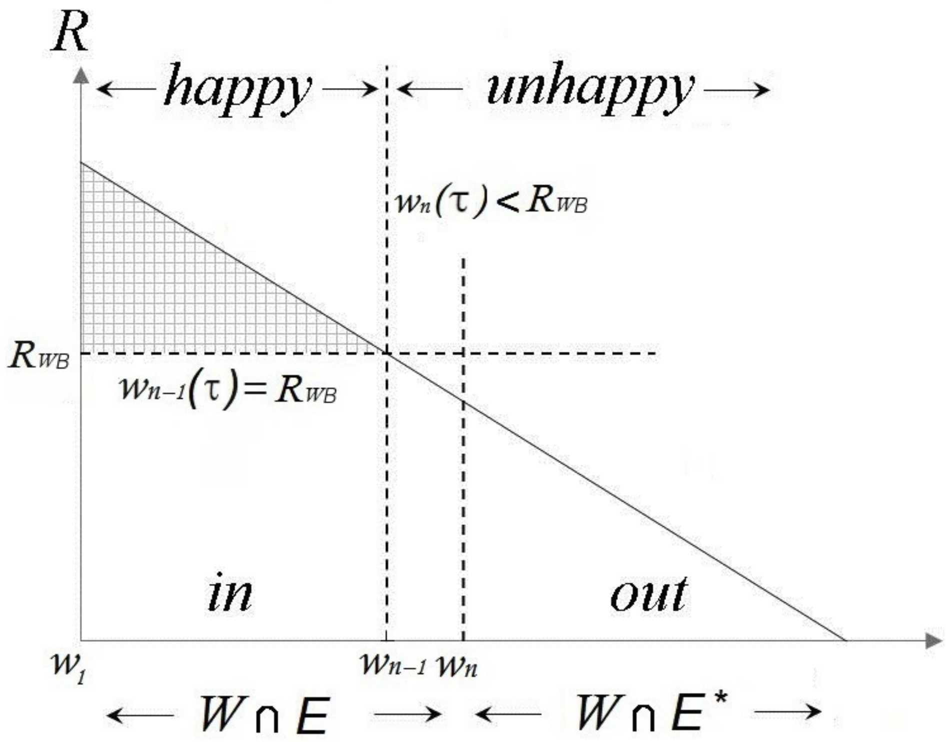


Figure 5.6: Movement rules for the W population derived from Schelling's description (see Chapter 5.1.10). 'Unhappy' agents lie to the right of the hatched area. The ratio R_{BW} causes one agent (w_n) inside to be 'unhappy', so w_n leaves the neighbourhood E. This decreases $W \cap E$ by one, thereby increasing R_{BW} so potentially making more W agents 'unhappy', and so on. The B population behaves in an identical manner.

Algorithm 6 Bounded Neighbourhood Model $\mathcal{I}$ Initialisation/Iteration Operator

```
Create population of agents B ∪ W
apply τ to agents
set W ∩ E and B ∩ E agents state = [in]
execute Algorithm 7
```

Algorithm 7 Agent w_n 'happiness' calculation: one timestep

```
calculate neighbourhood ratio R_BW
for each agent w ∈ W do
  if w_n(τ) < R_BW then
    set w_n [happy]
  else
    set w_n[unhappy]
  end if
end for
execute Algorithm 8
```

any number of simulations, yet specific enough to be easily reproduced. In addition, by laying out the dynamics of the model, a deeper understanding of the model workings is gained. Here we present the Platform Model of Schelling's Bounded Neighbourhood Model in pseudo-code, consistent with algorithms 4 and 5, which are used as the basis for algorithms 6 and 8. The simulation initialises by creating the agents, applying a type and a tolerance, then sorting them into neighbourhoods (algorithm 6). Following this, agents calculate the ratio according to equation (5.3). By checking their tolerance against the ratio, agents are able to calculate if they are 'happy' and set their state appropriately. Once calculated the simulation selects the most tolerant 'happy' agent in E^* and moves them to E. If there are no 'happy' agents in E^*, the simulation selects the least tolerant 'unhappy' agent in E (algorithm 8). The movement rule is shown in algorithm 8.

5.3 Simulation Platform

The Simulation Platform is implemented in Netlogo [3], a widely used simulation language developed at Northwestern University. Netlogo is one of the platforms used by the Computational Social Science Society of the Americas (CSSSA). Netlogo's language is based on Java and is universally compilable. It allows a simple translation of pseudo-code into machine readable code. The code is structured around agents called turtles, patches or links. These can be interconnected in a number of ways to produce a vast array of models. To simulate Schelling's Bounded Neighbourhood Model, Netlogo can be coded to represent the

Algorithm 8 Agent w_n movement rule: one timestep

if $\exists w_n \in (W \cap E^*)$ with $w_n(\tau) \geq R_{BW}$ **then**
  move w_n with max τ to E {(a) happiest outside moves in}
else
  if $\exists w_n \in (W \cap E)$ with $w_n(\tau) < R_{BW}$ **then**
    set w_n with min τ to E^* {(b) unhappiest inside moves out}
  end if
end if

model using turtles (for agents) and patches (for location). The code is translated from the Platform Model and implemented producing the Simulation Platform. The use of the Netlogo commands `max-one-of` and `min-one-of` allows the selection of agents in order of their tolerance.

Changes to the model parameters are tested using the `BehaviourSpace` tool. This tool automates the variation of system parameters, allowing the simulation to systematically run through vast numbers of initial conditions. This means the testing of Schelling's model over all 5000 initial population configurations (from $(w_n = 1, b_n = 1)$ to $(w_n = 100, b_n = 50)$) can be done easily, and without the need for extra code. Because Schelling's model is deterministic, it is unnecessary to run each initial condition more than once, since the same initial condition will return the same result. The outputs are presented in a spreadsheet which submits easily to analysis. Once the model reaches a steady state (ı.e no more agents wish to move) the number of iterations (I) is recorded and the simulation terminates. However a significant drawback of the `BehaviourSpace` tool is the results from the simulation are not outputted until the entire experiment has finished. This causes two major issues: firstly, if the simulation crashes before the experiment ends, all the data is lost. Debugging and verifying the code becomes even more vital. Secondly, as the experimental data increases with each run, the simulation begins to struggle with the load. This means that if the experiment is too big the system will freeze. For this reason the number of iterations allowed in each simulation run was limited. This limit created artefacts in Chapters 7 and 8 by halting runs which had not reached equilibrium. These were eventually resolved by increasing the limit but reducing the number of runs. It is important to note that the limitations of Netlogo are outweighed by the usefulness of the tools it provides. The `BehaviourSpace` environment, although limited, can be adjusted to cope with heavier workloads.

5.4 Results Model

In keeping with the CoSMoS process described in Chapter 4.1, the results of the simulation will be measured and analysed using hypothesis testing (see Chapter 4.4). To recover Schelling's flows, the population level will be recorded at each iteration and plotted on a two dimensional diagram, identical to Schelling's results. One iteration is defined as the movement of one agent, as described in Chapter 5.1.10. A run terminates when no more agents wish to move, with I being the number of iterations taken to reach that point. The configuration of the final population is calculated using M (see Equation (4.1)). In later experiments, these results will be used to test the hypothesis, that changing the parameters of the model changes the results.

5.4.1 Validation

By comparing the results of the simulation with Schelling's results, evidence is provided towards validation of the simulation. Because Schelling's movement rules were imprecise, we are unable to measure the number of iterations the model takes to reach a steady state. Instead the flows of the populations are plotted from each initial condition. Figure 5.7 shows a comparison of these simulation flows, against Schelling's diagrammatical flows. The flows of the simulation mimic Schelling's results, the parabolas clearly identify identical boundaries.

5.5 Summary

This chapter has presented a formalisation of Schelling's Bounded Neighbourhood Model within the CoSMoS framework (Section 5.1), from which a Netlogo simulation has been developed (Section 5.2). The formalisation and development of a simulation allows for the testing of the effects of a number of possible parameter values of the system. This initial work has focused on the formalisation of the model and the development of a simulation. The analysis of results (Section 5.4) show that the simulation replicates Schelling's population dynamics. However there is a small ambiguity, the B population in Schelling's model appear, from his diagrams, to exit the neighbourhood more slowly than the simulation results. This is an interesting point, and is explored further in Chapter 8.

Schelling's Bounded Neighbourhood Model can be considered a highly simplistic first step in the understanding of how prejudices can effect the movement of populations into and out of 'neighbourhoods'. The simplicity of the model allows a full understanding of the dynamics that are driving it but, unfortunately, those dynamics are unable to produce any 'interesting' results. However it is possible to use this model as a basis for a more

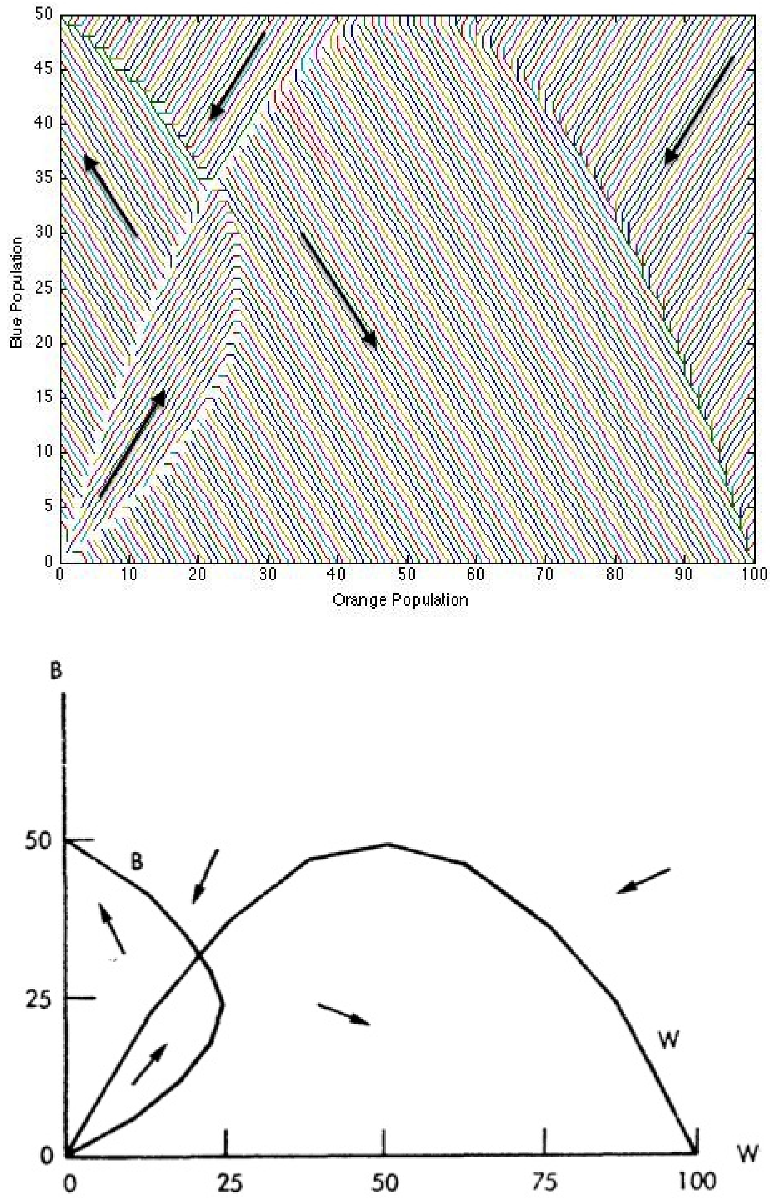


Figure 5.7: To validate the simulation a comparison of the simulation output (top) and Schelling's results (bottom) is made. Schelling's boundaries can be seen in the simulation results with the flows following Schelling's results. However, the two right most arrows suggest that Schelling calculates a much slower exit of the B population.

complex model by modifying or adding appropriate features which allow the modified model to be tested systematically. Indeed, by developing a valid simulation of Schelling's Bounded Neighbourhood model, it is now possible to ask questions of the model, such as the existence of stable equilibria, that have previously been difficult to answer. This is done for the formalised Bounded Neighbourhood model to test the robustness of the assumptions made by Schelling (listed below) and carried out in Chapters 6, 7, 8 and 9.

Whose move is it anyway? The movement of the agents depends on their knowledge of every other agent's 'happiness'. By changing the movement rule, Chapter 6, explores the effect of randomly choosing agents who wish to move, rather than moving by order.

Who do you know? The fact that agents have complete knowledge of the neighbourhood. This is tested in Chapter 7, in which agents sample the neighbourhood to calculate their 'happiness'.

Should I stay, or should I go? This refers to the assumption that agents move as soon as they are 'unhappy'. This is tested in Chapter 8, where it is assumed their decision to move is based on a level of 'happiness' derived from the neighbourhood ratio.

Friends and neighbours. The fact that all agents are treated equally within type. By recasting Schelling's Bounded Neighbourhood Model as a network in Chapter 9, we explore the effects of social ties within the neighbourhood.

At each step the formalisation of Schelling's Bounded Neighbourhood Model is adjusted to develop a simulation to test these assumptions. The work begins with the movement of the agents and what effect their order of movement has on the results.

Chapter 6

Whose move is it anyway?

> "It is the least tolerant whites that move out first, and the most tolerant that move in first, and similarly for blacks." [74, p168]

6.1 Introduction

The formalisation of the Bounded Neighbourhood Model has highlighted a testable assumption. The movement of the agents is dependent on their complete knowledge of the 'happiness' of every other agent, as the least 'happy' inside (or the most 'happy' outside) moves first. By removing this knowledge, we can test the hypothesis that the movement of agents, ordered by their level of tolerance, has no significant effect on the results. In this chapter, we apply a variation to the agents' movement rule, and test it against Schelling's rule. Thus, we have two movement rules:

1. Agents move one at a time according to level of tolerance (as described by Schelling).

2. Agents wishing to move are chosen randomly.

6.2 Hypothesis

The hypotheses tests the effects of changing the movement rule on the model. We offer the null hypothesis (H_0) that changing the movement will have no effect on the outcome of the simulation. This is tested using two measures, static and dynamic (discussed in Section 4.4).

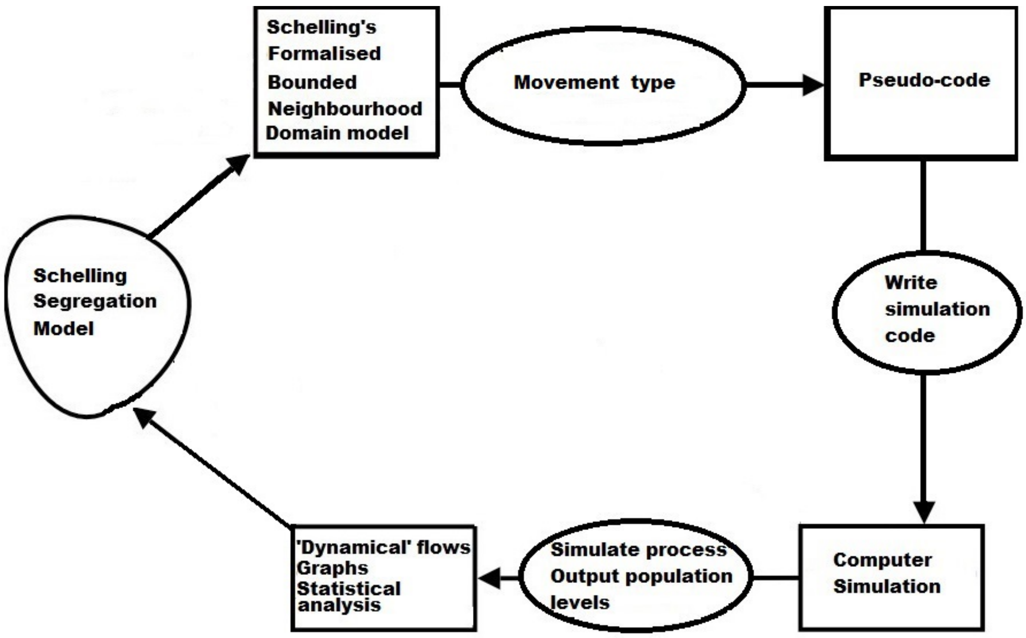


Figure 6.1: An application of the CoSMoS process in testing the effect of movement in Schelling's Bounded Neighbourhood Model. The movement rules are implemented in the Domain model. The effects of the two rules are tested using the tests described in Section 4.4. The results from the analysis feedback into Schelling's Segregation Model offering further insights into its mechanics.

6.2.1 Static measure

The static measure of the final population inside the neighbourhood, shows whether the alteration to the model produces stable mixed populations. Following equation (4.1), M is used to measure the final population configurations. The hypothesis, that changing the movement rule will produce mixed populations, is tested by comparing M to Schelling's findings of $M = 0$. A result of $M > 0$ suggests mixed populations have arisen, counter to Schelling's model.

6.2.2 Dynamic measure

The analysis of the population flows focuses on I, which is the number of iterations taken to reach a steady state. The system has reached a steady state when no more agents wish to move. The results are analysed using the Mann-Whitney-Wilcoxon U test and the Vargha-Delaney A-statistic, described in Section 4.4.

6.3 Domain model: changing the movement order

The Domain Model has two changes from Section 5.1. The major difference is that agent movement can be either stochastic or deterministic. The change to the agent movement rule is shown in the Platform Model (Section 6.4).

6.4 Platform Model

Using the parameters defined in Chapter 5, a Platform Model is developed using pseudo-code. Rule 1 is unchanged from algorithm 8. Rule 2, that agents move at random, can be accommodated with a small change to algorithm 8 to give algorithm 9. A minor change is made to the colour of the agents. The reason for this change is that some results are displayed using a colour code, using black and white colours would be difficult to display. Agents are now either orange $o_n \in O$ or blue (remains $b_n \in B$), with the ratios similarly adjusted giving R_{OB} and R_{BO}.

6.5 Simulation Platform

The Simulation Platform follows the same process as Section 5.3, with 5000 initial conditions covered. Since a stochastic process has been introduced, each initial condition is run using 25 different seeds. This helps protect against artificial results, that could be produced by the

Algorithm 9 Agent type o movement rule: one timestep

if $\exists o_n \in (O \cap E^*)$ with $o_n(\tau) \geq R_{OB}$ **then**
  move o_n with $o_n(\tau) \geq R_{OB}$ {(a) random 'happy' outside moves in}
else
  if $\exists o_n \in (O \cap E)$ with $o_n(\tau) < R_{OB}$ **then**
    move o_n with $o_n(\tau) < R_{OB}$ to E^* {(b) random 'sad' inside moves out}
  end if
end if

randomness in the model. The Netlogo command `one-of` allows the random selection of a single agents from a group. Thus, the change in the movement rule is a simple alteration to the movement algorithm.

6.6 Results Model

The results of the static analysis, M, is calculated from each initial condition (see equation (4.1)). The colour of the final populations from each initial condition, is mapped onto a two dimensional plane (as described in Section 4.4). This means that each point returns either orange or blue, depending on the final population configuration, giving a much clearer visual indication as to the more dominant population. Since Schelling's model is deterministic the results display a clear boundary for the populations. The random movement produces 25 different results, which are collated onto a single colour map (bottom Figure 6.2). In this case, the results from each seed show that, Schelling's clearly defined boundary has blurred into a 'region of uncertainty', insofar as it is unclear which population will dominate from the initial condition. In addition there is an increase in the dominance of the Orange population. Initial conditions that previously were ending with Blue populations, are now Orange. This can be seen by the change in the boundary on the random results. However, all the final populations are all still segregated, shown by $M = 0$. The dynamic results, are again displayed on a two dimensional chart identical to Schelling's. However, the results from the random movement are highly chaotic (see Figure 6.3). The flows that were clear in the original simulation, have all but disappeared.

By focusing on four initial conditions and comparing different seeds, we get a clearer indication of the population flows given by the random movement rule (Figure 6.5).

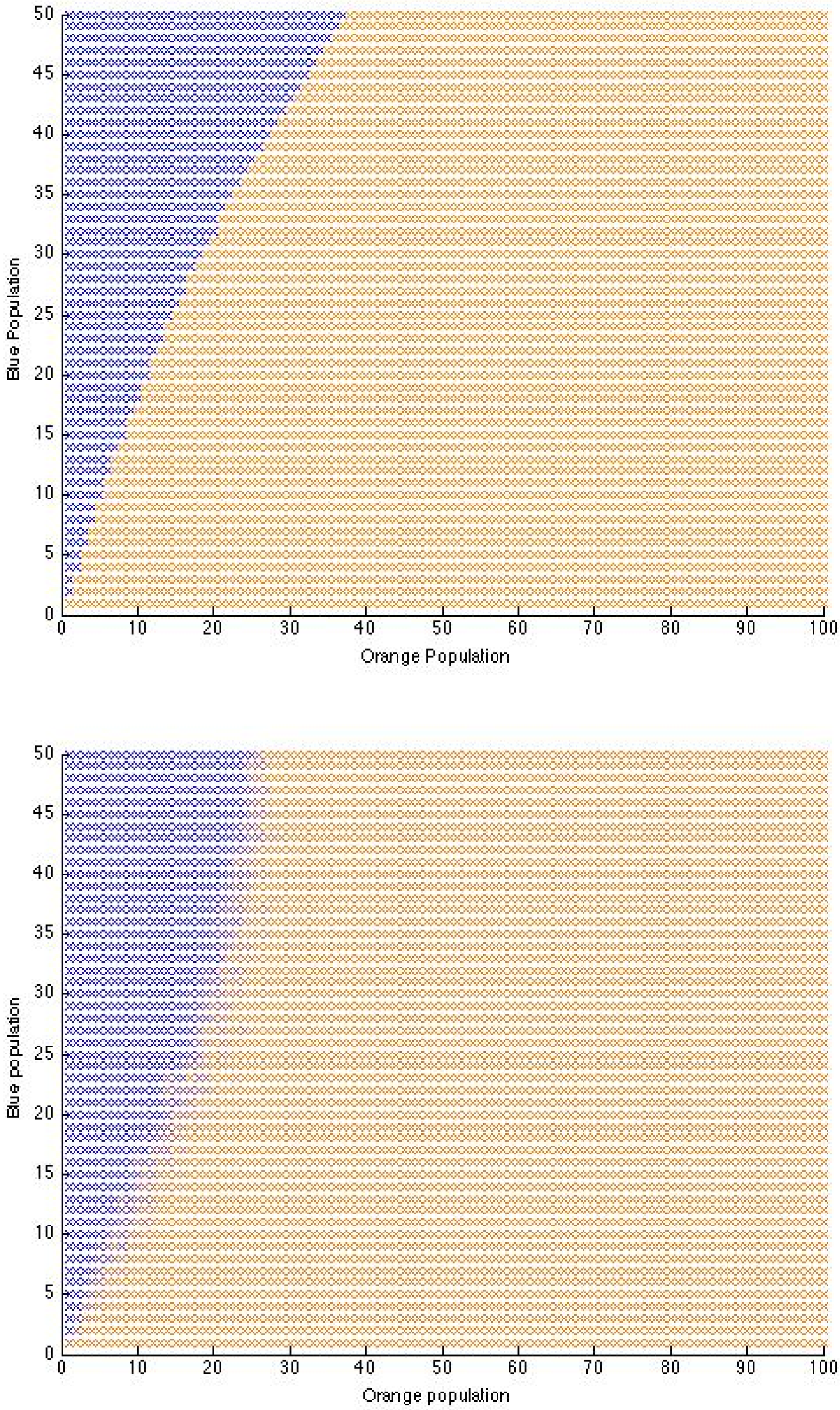
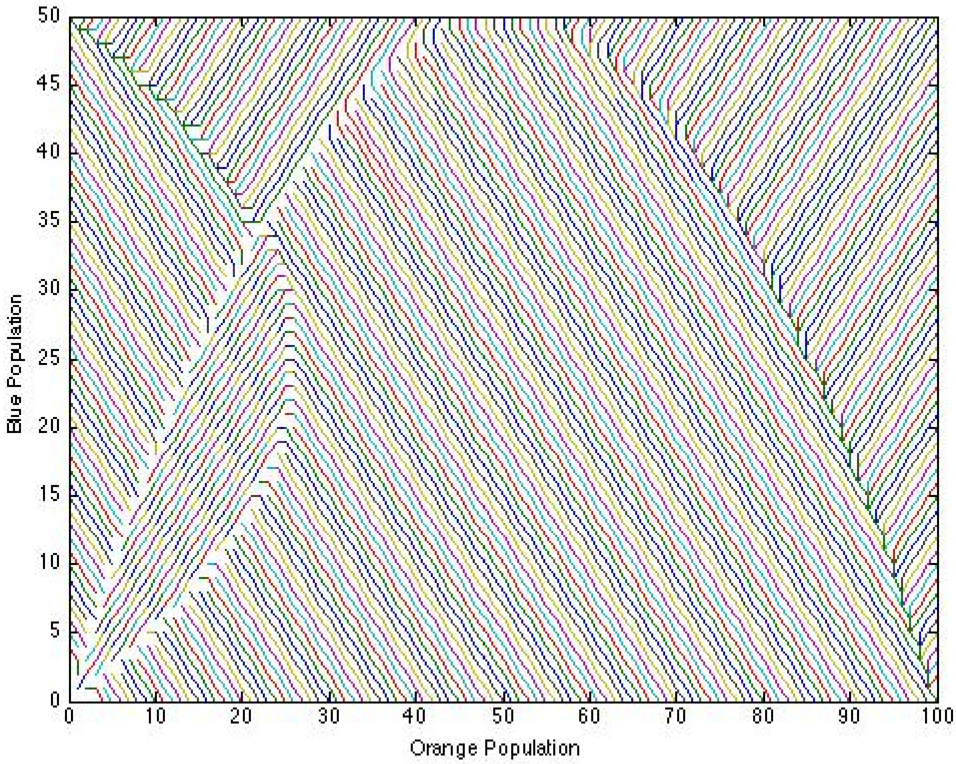


Figure 6.2: Colour coded comparison of deterministic (top) and random (bottom) movement. Coordinates correspond to the initial condition, whilst the colour represents the final population configuration. The deterministic results show a clear division between the populations, and that no mixed population steady states appear. The stochastic results are collated onto the colour map. The previously clear division is now a region of uncertainty. However the uncertainty does not lead to mixed populations.



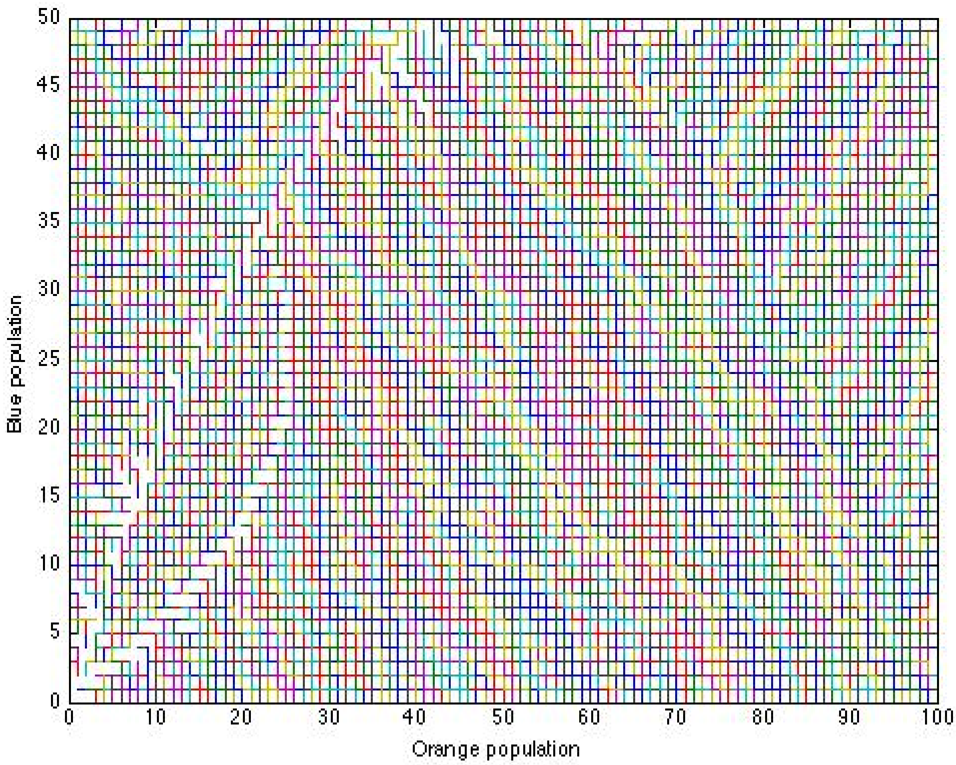


Figure 6.3: The population flow of the random movement rule, plotted on a two dimensional plane (bottom). The stochastic nature of agent movement partially obscures the parabolas that are clearly shown in the deterministic model (top).

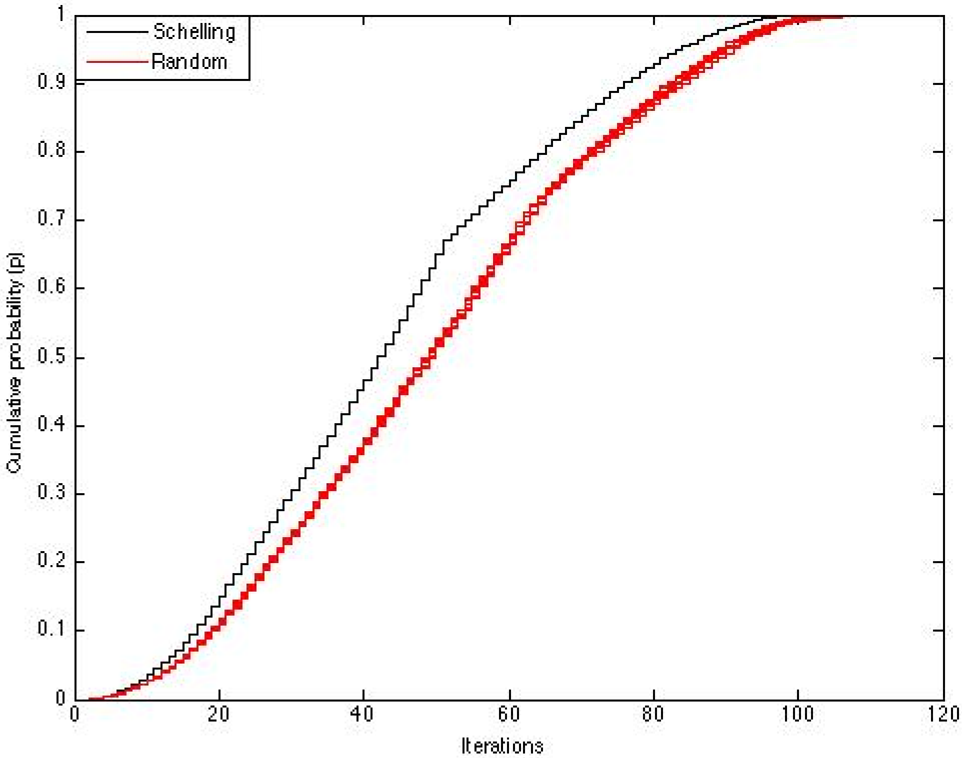


Figure 6.4: Cumulative frequency distribution of I from deterministic (*black*) and 25 random (*red*) results. The shift of the distribution to the right suggests that, random movement takes longer to reach a steady state. This evidence is supported by the results of the U-test, which calculates a significant difference $p < 0.005$ between the median of the deterministic model (I median $= 42$) and the random (I median $= 49$.

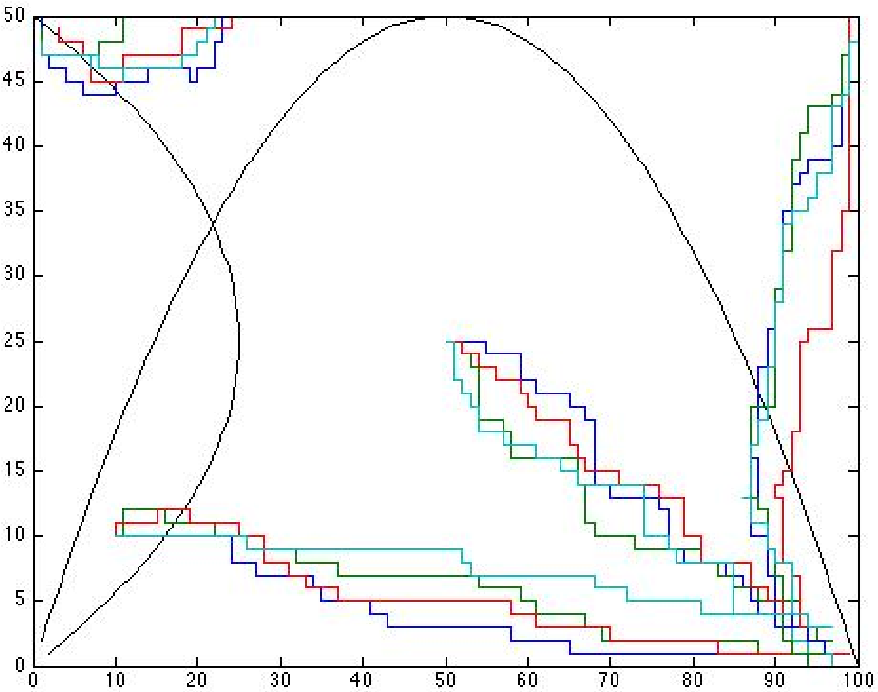


Figure 6.5: An edited version of the random movement rule. By focusing on four initial conditions, $(10O, 10B), (50O, 25B), (25O, 50B)$ and $(100O, 50B)$, a clearer picture of the population flows can be seen. Even with stochastic movement the agents still follow the population flows of Schelling's model.

6.6.1 Testing the hypotheses

The results from the simulation, are used to test the hypothesis that, changing the movement rule changes the results of the model. The null hypothesis H_0, that there is no change in the model, is tested using two measures, static and dynamic.

Static analysis

The static results are tested by measuring the value of M (see equation (4.1)). Since M is calculated for each initial condition, each seed returns 5000 values of M. By simply averaging the results across the seeds, a value of $0 \leq M \leq 1$, for each initial condition is returned. Values of $M > 0$ means that the simulation has produced mixed populations. However, the analysis of the final population levels shows $M = 0$. This means that none of the final states contained mixed populations. Thus, we cannot reject the null hypothesis, that randomising the movement of the agents has no effect on M. What the results fail to show is the change, from Schelling's clear boundaries, to the creation of a 'region of uncertainty' along the former boundary. Within this area the population could end up as either O or B, but not both.

Dynamic analysis

The dynamic results are tested using the U test. This is used to analyse I, the number of iterations taken for the simulation to reach a steady state. The deterministic nature of Schelling's model, means each initial condition produces a single value for I. This is compared to the corresponding value from the 25 random results. The results are then used to test the null hypothesis, H_0, that the compared samples are drawn from the same population. The test calculates a statistic U that measures the rank of the data in the two samples. If $U_{obs} > U_{critical}$ the null hypothesis H_0 can be rejected. However, the sheer number of samples in this experiment increases the likelihood of a significant result. By measuring the effect size using the A test, it is possible to guard against this. In addition, the failure to reject H_0 does not imply proof of H_0; rejection is merely a proof by contradiction [84]. The results from the statistical analysis, with $n = 5000$, indicates a significant difference in I, between the deterministic (median $I = 42$) and random (median $I = 49$) results. Calculating the A statistic (see Section 4.4.3) shows a small effect size with $A_{12} > 0.56$. This can be clearly seen in Figure 6.4 and, thus, we can reject H_0.

The effect of the random movement rule can be further explored, by comparing values of I from the results of the 25 seeds. By focusing on four initial conditions (see Figure 6.5), and plotting their distributions we are able to see the variation in I (see Figure 7.9). The large

	Median I	p	A_{12}	M
Deterministic	41	1	-	0
Random	49	**0**	**0.5717**	0

Table 6.1: Results from the analysis of the Bounded Neighbourhood simulation.

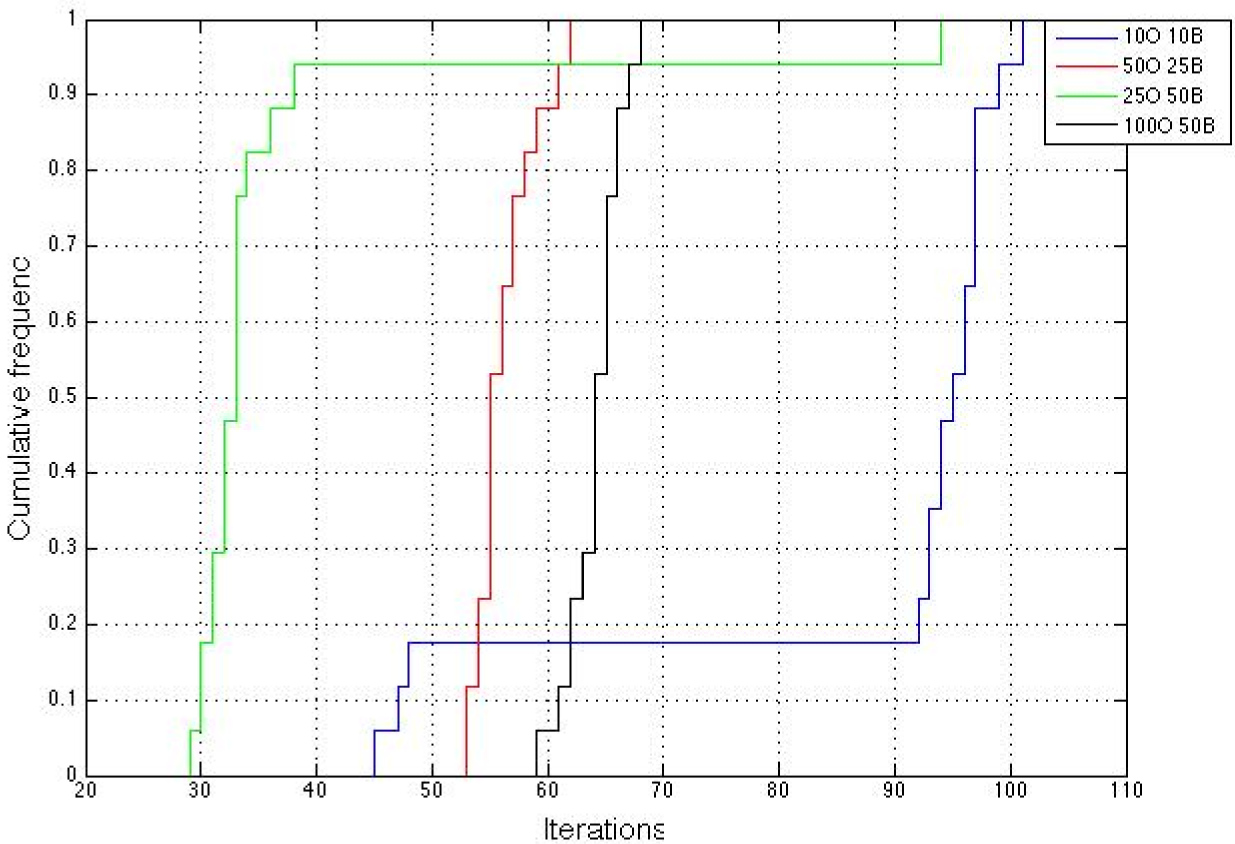


Figure 6.6: Cumulative frequency distribution of I from 25 different seeds. Populations $(10O, 10B)$ (blue), $(50O, 25B)$ (red), $(25O, 50B)$ (green) and $(100O, 50B)$ (black) are initialised and run 25 times to produce a distribution of runtimes. For the $(100O, 50B)$ and $(50O, 25B)$ results, there is little variation in I, meaning the populations always segregated into the same colour. However there is a much larger variation in the $(10O, 10B)$ and $(25O, 50B)$ results. This is due to the final populations being either all O or all B, depending on the seed.

variation in the distributions, for the initial conditions $(10O, 10B)$ and $(25O, 50B)$, can be attributed to the difference in the number of iterations taken to reach one of the two final states. Since there are less iterations to reach $50B$, compared to $100O$, initial conditions that reach both have a wider distribution. Since the random movement of an individual in a smaller populations has more of an effect, this result is unsurprising for the initial population $(10O, 10B)$. What is more interesting is the final population configuration, this is O around 80% of the time, as opposed to the expected 66%[1]. More surprising is the initial population $(25O, 50B)$. This population initialises at the maximum τ for the O population and, as such, should segregate to B. However the results indicate that, around 5% of the time the population segregates to O. Both these results highlight the increased dominance of the O population. However, when compared to results from the original simulation, none of the results returned a significant difference (see Table 6.2).

6.7 Conclusions

The work presented here, explored the effect of Schelling's deterministic movement rule. The results show that changing the the movement rule in the Bounded Neighbourhood Model, from ordered to random movement, causes a significant difference in the number of iterations (I) taken to reach stability. By applying the Vargha-Delaney test, we have shown the difference is not just a consequence of the number of samples used in analysis. However, the changes have no effect on the final population configuration of the model, as the only stable populations are totally segregated ($M = 0$). The ability to affect, what Schelling would call, "the process of occupation" [74, 170] but not the final segregation, highlights the model's resistance to change. In addition the change in the movement rules increased the dominance of the Orange population. In the original model, there was a definite divide of $R_{BO} = 2 : 1$ for $B < 25$. Any population initialising above that ratio would end up Blue, those below ended up Orange. However, the ratio steadily decreases as B increases, meaning when $B = 50, R_{BO} = 1.5 : 1$. In the random model, the divide becomes blurred and changes shape, so that, $R_{BO} \approx 2 : 1 \; \forall B$. This is because it is no longer the most unhappy moving (which, when $R_{BO} > 1$, is an Orange), instead one is chosen at random. This means that, when $B > 25$ and $R_{BO} \approx 2 : 1$, although a Blue is more likely to be 'happy', they are twice as likely to be chosen.

The distribution of the Schelling model shows a disjoint in the curve. The distribution suggests that, by the time 50 iterations have been reached the model has reached a steady

[1] which is expected because the O population is twice that of B. This means the population would be expected to dominate, from an equal initial condition, two-thirds of the time (or 66%).

	10O, 10B			
	Median I	p	A_{12}	M
Deterministic	91	1	-	0
Random	95	0.444	-	0
	50O, 25B			
	Median I	p	A_{12}	M
Deterministic	51	1	-	0
Random	55	0.111	-	0
	25O, 50B			
	Median I	p	A_{12}	M
Deterministic	26	1	-	0
Random	33	0.111	-	0
	100O, 50B			
	Median I	p	A_{12}	M
Deterministic	51	1	-	0
Random	64	0.08	-	0

Table 6.2: Results from 25 different random seeds, taken from four initial condition. Although the results from all 5000 initial conditions returns a significant difference, there is no significant difference between the rules, from these initial conditions.

state 66% of the time. This is due to the disparity in population sizes, since by 50 iterations the entire B population will have made their moves, but only half of the O. This means that, although the majority population dominate the final mix, the minority population reach their steady state more quickly. This can be seen when exploring the results from different seeds from certain initial conditions. Here the distributions of I, for certain initial conditions, were stretched by the final configurations being either all O or all B. Further examination suggests that randomising the movement of the agents, increases the dominance of the majority population. However, no significant difference was found in the runtimes from the four selected initial conditions. This suggests that the significant difference in the original results was an artefact of the sample size. However, the results of the distributions (Figure 7.9) show a similar increase compared to the cumulative results in Figure 6.4. This means none of the random results reach stability before the expected result. Whilst this is not completely unsurprising, it highlights the simplicity of the model. Since, with a more complex model, one could expect a random results to perform at least as well at least once, compared to the control. The overall results suggest that, whilst the ordered movement of agents quickens the process of segregation, there is no discernible effect on the segregation of the populations. This can be seen as altering the kinetics of physical processes, but not the thermodynamics. It is clear that, a much more significant alteration to the movement of agents is necessary to change the stability of the system. Simply randomising those who are leaving is not enough to change the results of segregation the model, however the random movement can change the colour of the final population.

Further work could explore increasing the uncertainty in the model, by choosing from all the agents at random, rather than from those who wish to move.. Even so, the experiment has shown that the segregation is not driven simply by the order of agent movement. Although this might seem an obvious statement, it is important to be able to provide evidence for it. In addition, this has been a successful application of the CoSMoS process to Schelling's model. Continuing the process, the results from the simulation are fed back into the Schelling Segregation model, increasing understanding, and raising new questions.

Chapter 7

Who do you know?

> "Information is perfect: everybody knows the color ratio within the area at the moment he makes his choice." [74, p168]

7.1 Introduction

In Schelling's Bounded Neighbourhood Model, agents have a precise knowledge of the colour ratio of the neighbourhood. From this knowledge they make a decision about moving. Previous work (Chapter 6) has shown that, although changing the order of movement increases the number of iterations taken to reach a steady state, the final outcome is always complete segregation. In this chapter, rather than calculating the ratio from the neighbourhood in its entirety, agents select a sample of other agents in the neighbourhood, and use that to calculate their ratio. This means that agents no longer have a uniform response to the environment.

7.2 Hypothesis

Consistent with Chapter 6, the null hypothesis (H_0) is that changing the agents' perception will have no effect on the results of the simulation. This is tested using the two familiar measures, static and dynamic.

7.2.1 Static measure

The static measure M is used to analyse the mix of the final population. M is calculated according to equation (4.1), and compared to the mix of $M = 0$, resulting from Schelling's model. Values of $M > 0$ indicate mixed populations and highlight a change in the results.

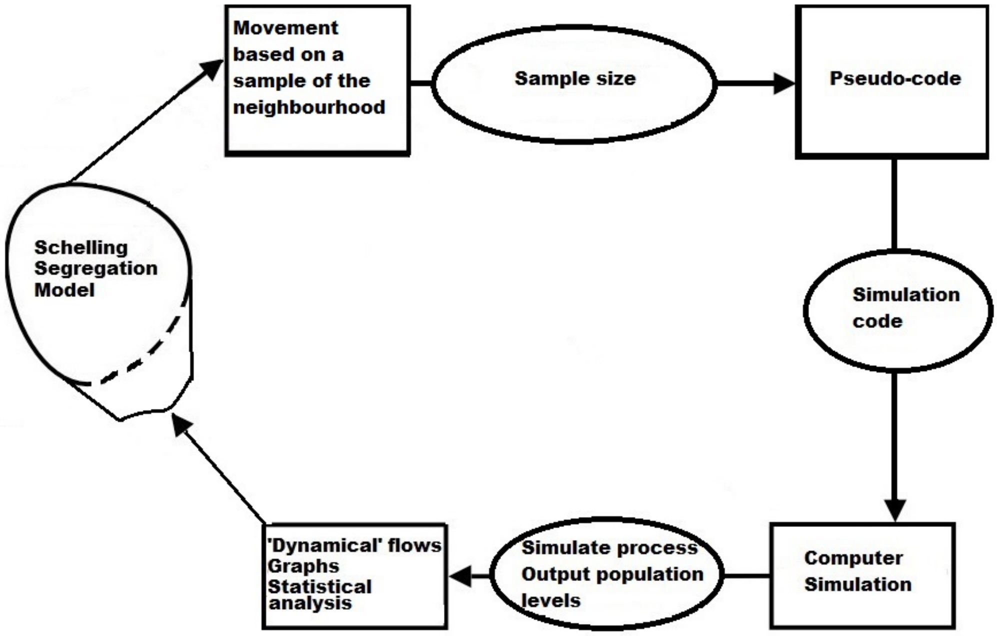


Figure 7.1: An application of the CoSMoS process in testing the effect of agents' exact knowledge of the ratio in Schelling's Bounded Neighbourhood Model. Movement based on the ratio of a sample of the neighbourhood is added to the Domain Model. The size of the sample becomes an adjustable parameter, with the effects of the adjustment tested as previously. The results feed back into Schelling's segregation model, this time offering different insights.

7.2.2 Dynamic measure

The dynamic measure tests whether the changes in the model, effects the population flows. This is tested by measuring the parameter I (described in Section 5.4), and comparing it to the results from the original simulation. The results are analysed using the Mann-Whitney-Wilcoxon U test and (if necessary) the Vargha-Delaney A statistic (described in Section 4.4.5).

7.3 Domain model: sampling the neighbourhood

The language of the Domain Model is retained from the formalisation presented in Chapter 5. Two populations of agents $o_n \in O$ and $b_n \in B$, calculate the ratio of the neighbourhood E (R_{BO} and R_{OB} respectively). Agents compare R against their tolerance τ. If the comparison is unfavourable, $o_n(\tau) < R_{BO}$ or $b_n(\tau) < R_{OB}$ the agent leaves the neighbourhood.

7.3.1 Changes to the Domain Model

In this extension agents take a random sample (S) of the neighbourhood (E) and use it to calculate a sample ratio. This gives the ratio for $o_n \in O$ population as

$$R_{BO^S} = \frac{\#(B \cap S)}{\#(O \cap S)}, \tag{7.1}$$

with an equivalent equation for b_n of

$$R_{OB^S} = \frac{\#(O \cap S)}{\#(B \cap S)}. \tag{7.2}$$

It is useful that, as S approaches E, the model tends to Schelling's Bounded Neighbourhood Model, since this can be used to validate the simulation.

7.4 Platform Model

The implementation of the alteration to the model is shown in algorithms 10 and 11. The algorithms show the process for an agent o_n, an equivalent process governs the b_n agents. The only change to the algorithms presented in Section 5.2 is the calculation of R in algorithm 8. Creating a sample ratio from the total neighbourhood is a simple change in how the variables used to calculate the ratio are ascertained. Now agents take a random sample (S) of the neighbourhood and use it to calculate a sample ratio R_{OB^S} or R_{BO^S}. This new ratio calculation propagates through the Platform Model, changing R_{OB} and R_{BO} to R_{OB^S} and R_{BO^S}

Algorithm 10 Agent type o 'happiness' calculation: one timestep

calculate sample ratio R_{BO^S}
while $o_n(\tau) < R_{BO^S}$ **do**
  for each agent $o \in O$ **do**
    if $o_n(\tau) < R_{BO^S}$ **then**
      set o_n [happy]
    else
      set o_n[unhappy]
    end if
  end for
  execute Algorithm 11
end while

Algorithm 11 Agent type w movement rule: one timestep

if $\exists o_n \in (O \cap E^*)$ with $o_n(\tau) \geq R_{BO^S}$ **then**
  move o_n with max τ to E
else
  if $\exists o_n \in (O \cap E)$ with $o_n(\tau) < R_{BO^S}$ **then**
    set o_n with min τ to E^*
  end if
end if

respectively. However, in other respects, the movement rule is unchanged from Schelling's original model, meaning agent's still move in order of tolerance (see algorithm 11).

7.5 Simulation Platform

The Platform Model is simulated using Netlogo (see Section 5.2). Using this tool maintains a consistency that is important for valid comparisons of results. In addition, Netlogo has a number of useful environments that help facilitate the experiment. For example, the BehaviourSpace environment automates the testing of a number of different parameters, over a range of values. This is useful for testing across all initial conditions, with varying values of S. For consistency, the initial population values are retained from previous experiments. This gives 5000 data points for each value of S. These are analysed to test the hypotheses using the techniques set out previously (see Section 4.4). The values of the sample size S are 150 (for validation) $75, 50, 25$ and 10. These values, whilst arbitrary in some aspects, nonetheless, offer a good sample representation of a range of possible sample sizes. Since each sample is chosen randomly, 25 different runs of each initial condition are performed, identical to Section 6.5.

7.6 Results Model

The results from the simulation are validated and analysed, using the static and dynamic measures. The results of which provide evidence towards accepting, or refuting, H_0.

7.6.1 Validation

The simulation is validated by setting $S = 150$ (the total number of agents). This makes the sample the complete neighbourhood. As with the previous validation (Section 5.4.1) the output is measured against Schelling's results. The analysis of the static results shows $M = 0$, whilst the colour maps are identical (Figure 7.2). The population flows are compared in Figure 7.3 and show identical dynamics.

Following the validation of the simulation the remaining values of S are analysed. The results are analysed using the static and dynamic measures. This allows testing of H_0, that the reduction of agents' knowledge has no effect on the model. For the analysis of the static results, M is calculated from the steady state, reached by each initial condition. The value of M is then compared to Schelling's findings of $M = 0$. The final populations from each of those initial conditions, are displayed as colour maps (Figures 7.4 and 7.5). The dynamic measure (I) is used to test for changes in the number of iterations the system takes to reach a steady state. To give a visual comparison to Schelling's work, the results are displayed as flows (Figures 7.3, 7.6 and 7.7). The dynamic results are tested using the U test (described in Section 4.4.3).

7.6.2 Static analysis

To test H_0, a comparison of the final population configurations is made. Results can be seen in Figures 7.4 and 7.5, as well as in Table 7.1. Final populations are represented by their colour, with their location on the grid representing the initial condition. As can be seen, as S is reduced the dominance of the O population is increased.

7.6.3 Dynamic analysis

The dynamic test of the hypothesis argues that changing S will have no effect on I. The population flows in Figures 7.6 and 7.7 show that, as the sample size is reduced, the flows of the populations are disrupted (see Figures 7.5 and 7.5).

A comparison of the number of iterations (I) to reach a steady state is shown in Figure 7.8, with a full analysis of results presented in table 7.1.

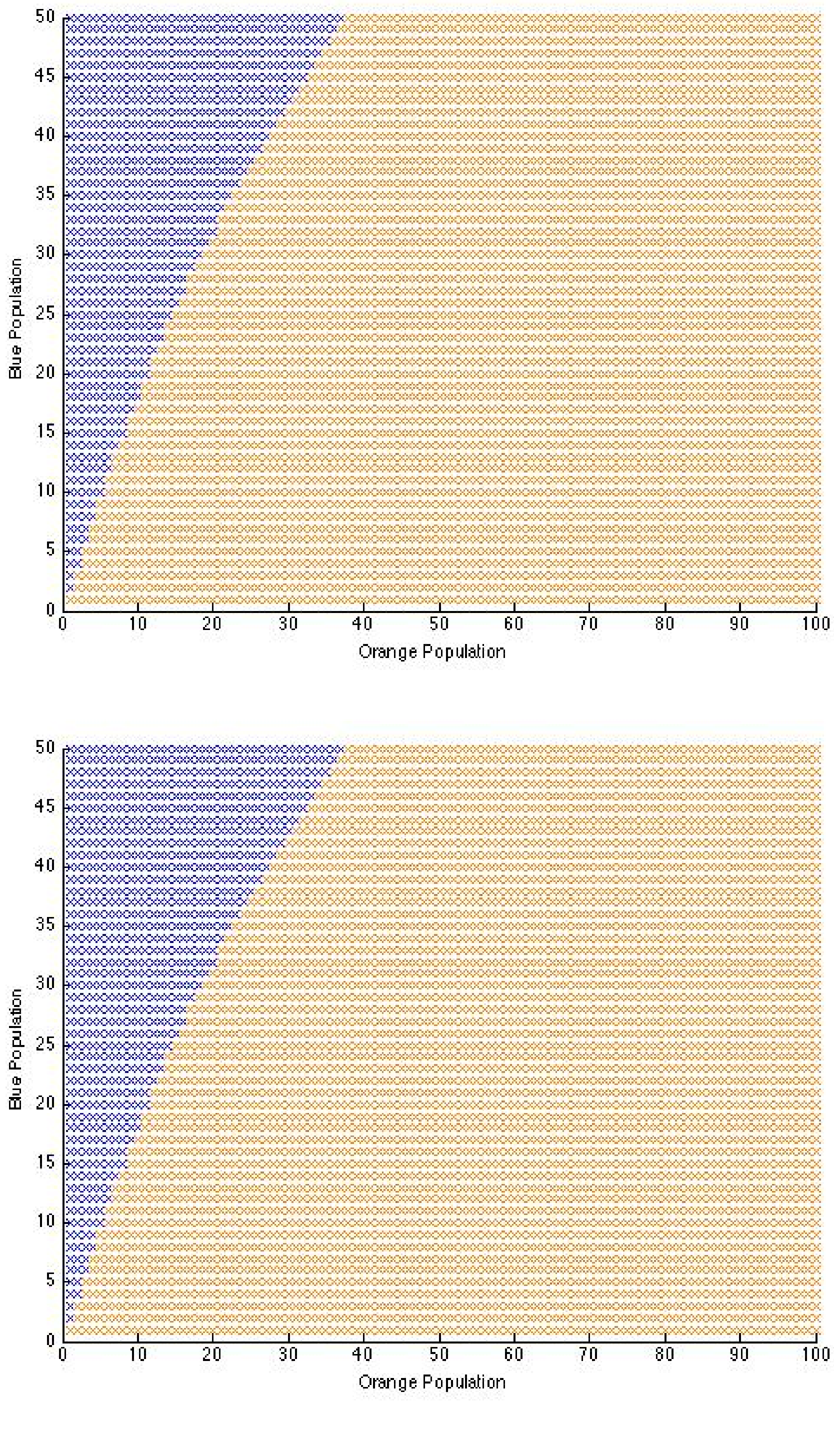
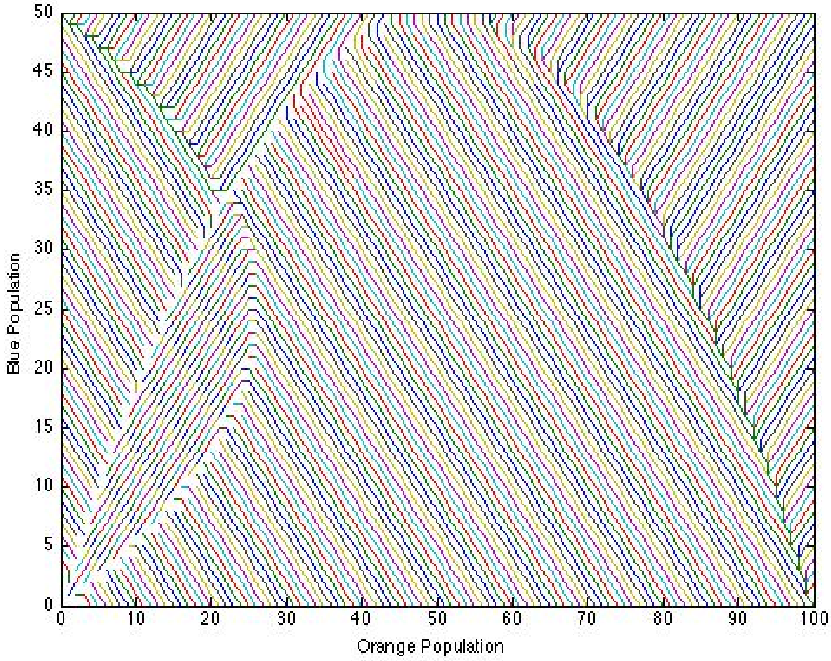


Figure 7.2: Comparison of the population colour maps from the validation results, for $S = 150$ (top), against the results from the Schelling Segregation model from Chapter 5 (*bottom)*.



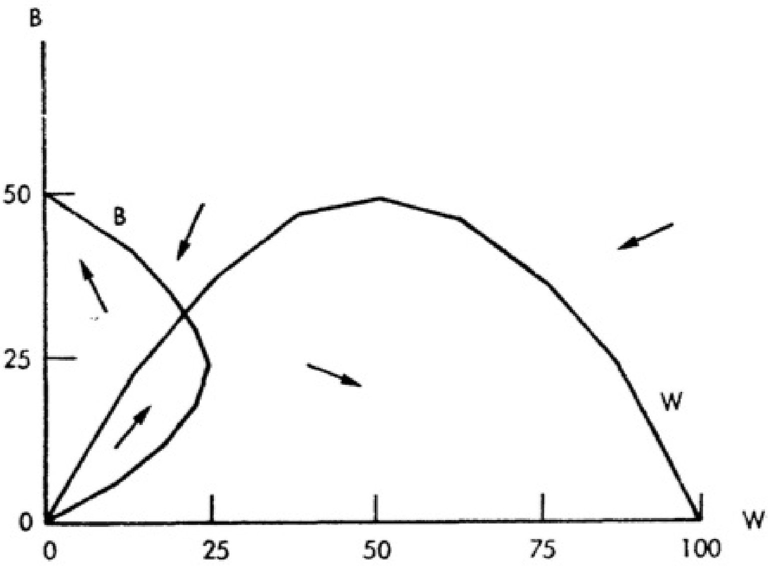


Figure 7.3: Comparison of the population flows of the validation results for $S = 150$ (top), against Schelling's results from [74] (*bottom)*.

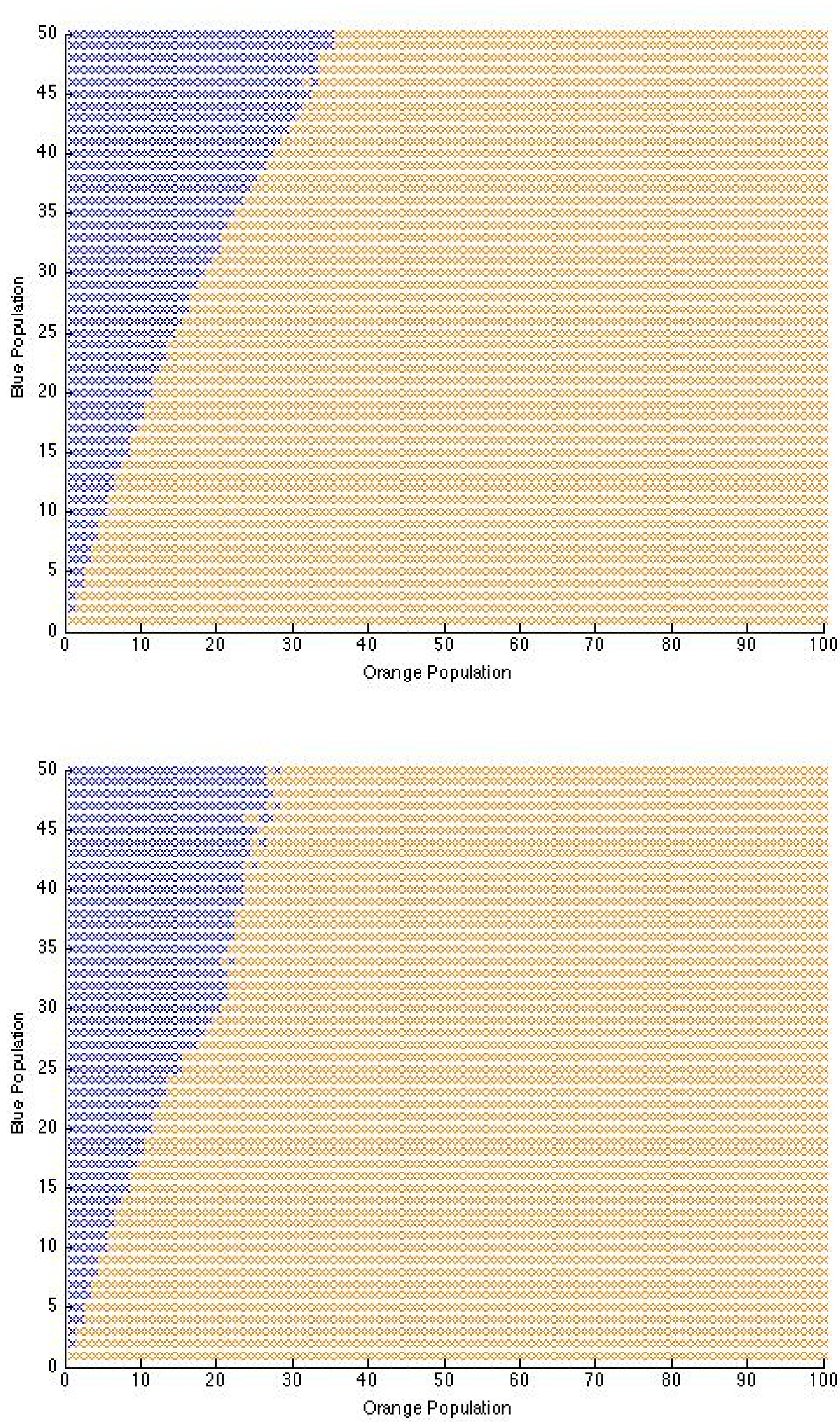
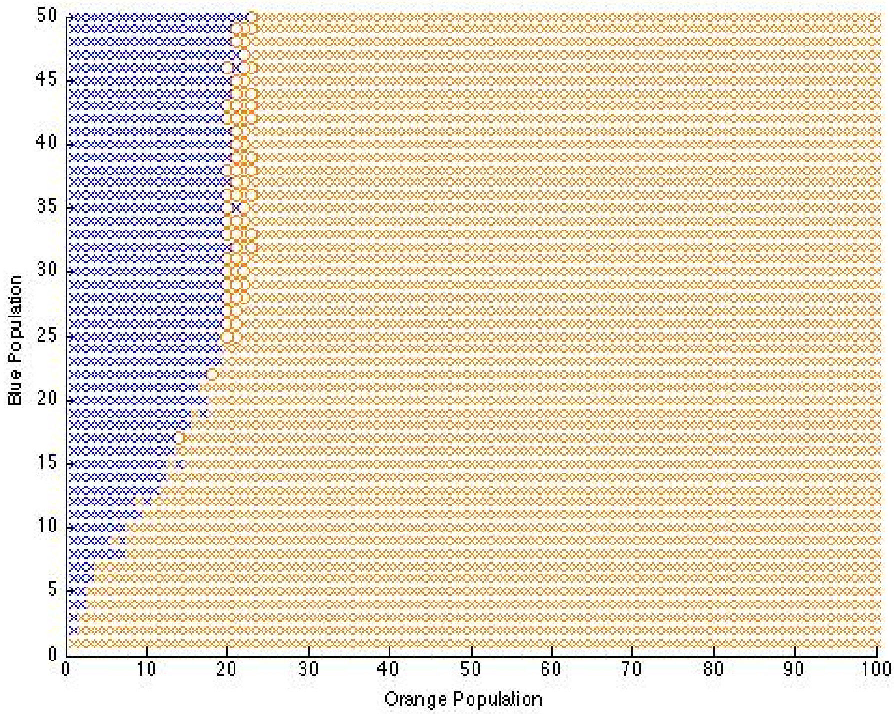


Figure 7.4: The colour coded comparison of final population configurations. Here a comparison is made between $S = 75$ (top) and $S = 50$ (bottom). Co-ordinates correspond to the initial condition, whilst the colour represents the final population configuration. The results show a clear division between the populations and no mixed populations.



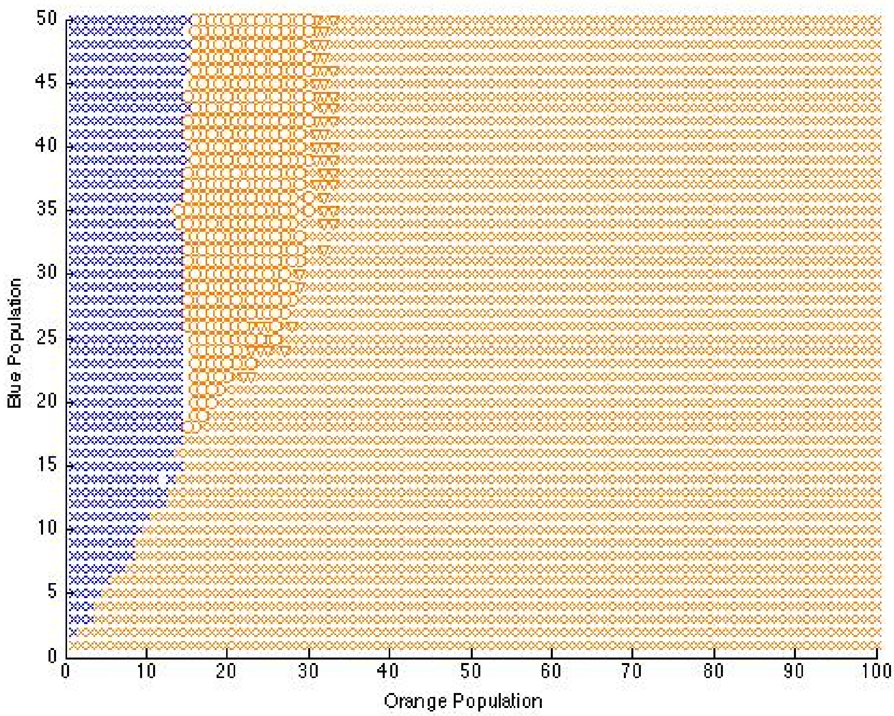


Figure 7.5: Colour coded comparison of final population configurations. Here a comparison is made between $S = 25$ (top) and $S = 10$ (bottom). Co-ordinates correspond to the initial condition, whilst the colour represents the final population configuration. The results show a clear division between the populations, and that although mixed populations appear (shown as circles), they have already 'tipped' (*i.e.*, have a ratio above $4:1$) and are in the process of segregating.

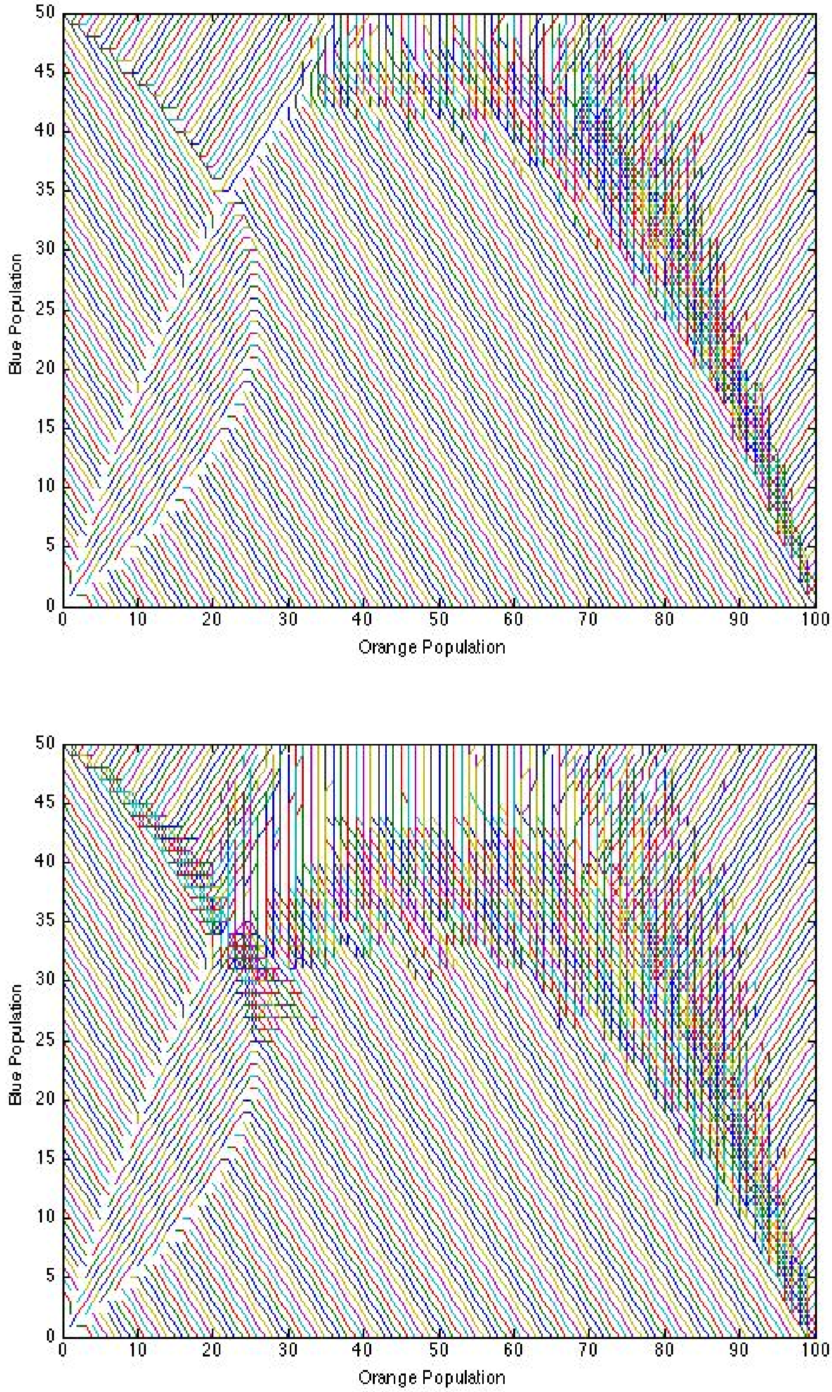
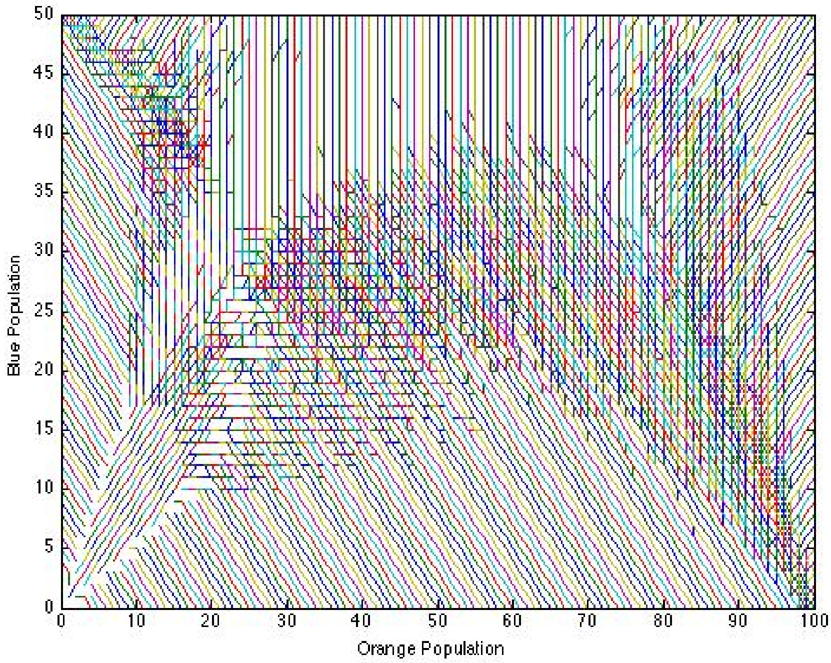


Figure 7.6: Comparison of the population flows for $S = 75$ (top) and $S = 50$ (bottom). When the population is larger than the sample size, the parabolas begin to distort. However, as the population numbers in the neighbourhood reduce, the simulation reverts to Schelling's results.



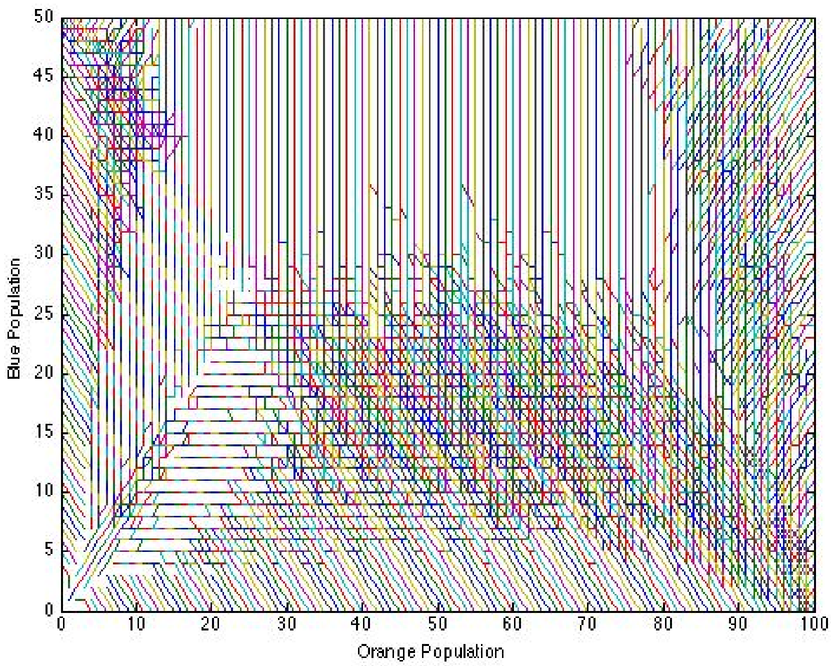


Figure 7.7: Comparison of the population flows for $S = 25$ (top) and $S = 10$ (bottom). When $S = 10$ the model is only able to replicate Schelling's flows for populations below 10.

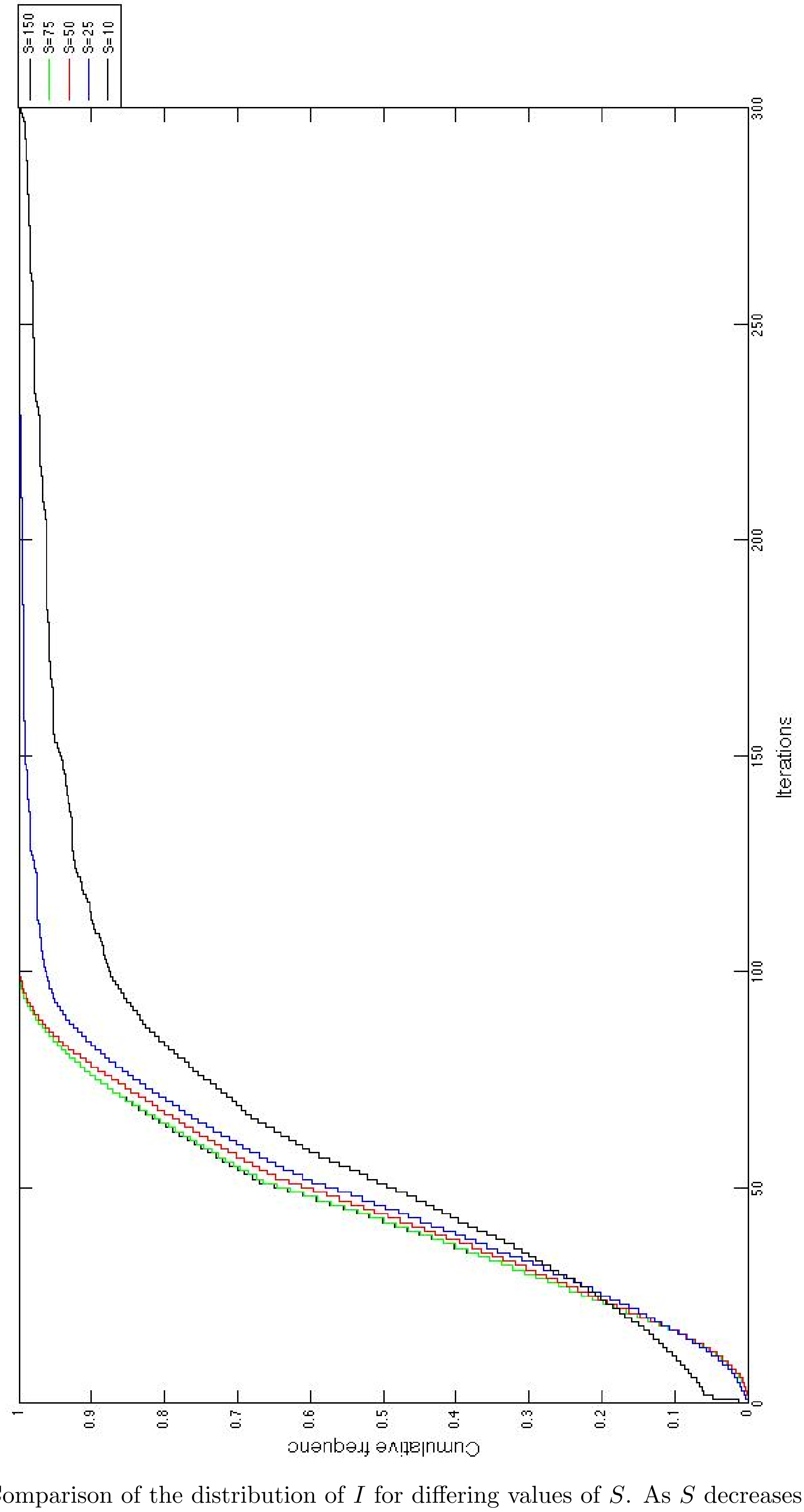


Figure 7.8: Comparison of the distribution of I for differing values of S. As S decreases, I increases, shown by the movement of the distributions to the right. The difference between $S = 150$ and $S = 75$ in negligible, shown by the almost overlapping distributions. There is a marked difference with $S = 10$, with only 85% of populations stabilising by 100 iterations. Indeed some populations took up to 300 iterations to stabilise.

S	Median I	p	A_{12}	M
150	41	1	-	0
75	42	0.82	-	0
50	43	< 0.05	0.518	0
25	45	< 0.05	0.559	0
10	50	< 0.05	**0.604**	0

Table 7.1: Results from different S values. The dynamic results show an increase in the median of I, which is significant for $S \leq 50$. However, only the smallest value of S produces a noticeable effect size, according to the Vargha-Delaney A statistic.

The results, displayed in Table 7.1, shows the disruption increases I as S is reduced; however, there is only a significant effect for $S = 10$. This can be seen in Figure 7.8, where the distribution of $S = 10$ is noticeably larger than the other values of S.

Focusing on a number of different seeds, allows further exploration of the results (displayed in Table 7.2). The results show that when $S = 10$ and the population initialises with $(50O, 25B)$ the value of I dramatically increases. Further exploration of I shows that, the system maintains a mixed population of around $25O$ and $35B$ for up to 700 iterations (see Figure 7.9). However, eventually, the system segregates to an all O population.

7.7 Conclusions

The work presented in this chapter introduces a different 'noise' into the system. Unlike Chapter 6, where the movement of the agent was randomised, here the random element was in their calculation of the ratio. For each sample size parameter change, the simulation was run 25 times to counter any possible artefacts from the stochasticity. The movement order remained unchanged, with the agents moving in order of their tolerance *assuming they are 'sad'*. As expected, when S was larger than the population in the neighbourhood, the results replicated Schelling's original model, with $M = 0$ and the median of $I = 41$. It was hoped that reducing the size of the sample size used to calculate the ratio might 'trick' agents into remaining, even when the actual ratio was above their tolerance. Unfortunately, results from the static analysis showed that $M = 0$. The results of the colour maps (Figures 7.4 and 7.5), show an interesting parallel with results from Chapter 6. Once again the, previously clear, division changes shape. The location of the change, however, is different for different sample sizes. When $S = 75$ the change is hardly noticeable, because most of the initial conditions

S	Median I	p	A_{12}	M
		$10O$,$10B$		
150	91	1	-	0
75	91	1	-	0
50	91	1	-	0
25	91	< 0.05	0.5001	0
10	97	< 0.05	0.53	0
		$50O$,$25B$		
150	51	1	-	0
75	51	1	-	0
50	51	1	-	0
25	51	< 0.05	0.5	0
10	65	< 0.05	**0.68**	0
		$25O$,$50B$		
150	26	1	-	0
75	26	1	-	0
50	26	1	-	0
25	46	< 0.05	0.53	0
10	428	< 0.05	$>$ **0.71**	0
		$100O$,$50B$		
150	51	1	-	0
75	51	1	-	0
50	52	0.3	-	0
25	52	0.114	-	0
10	57	< 0.05	$>$ **0.71**	0

Table 7.2: Results from 25 different random seeds, for the different values of S, are taken from four initial conditions. The results suggest that effects of the sample are only noticeable when the initial population is larger than the sample size. The largest variation is when $S = 10$ and the initial populations is $25O, 50B$. These conditions cause a dramatic increase in I, further investigation of the initial condition indicates, the system sustains mixed populations for hundreds of iterations (see Figure 7.9).

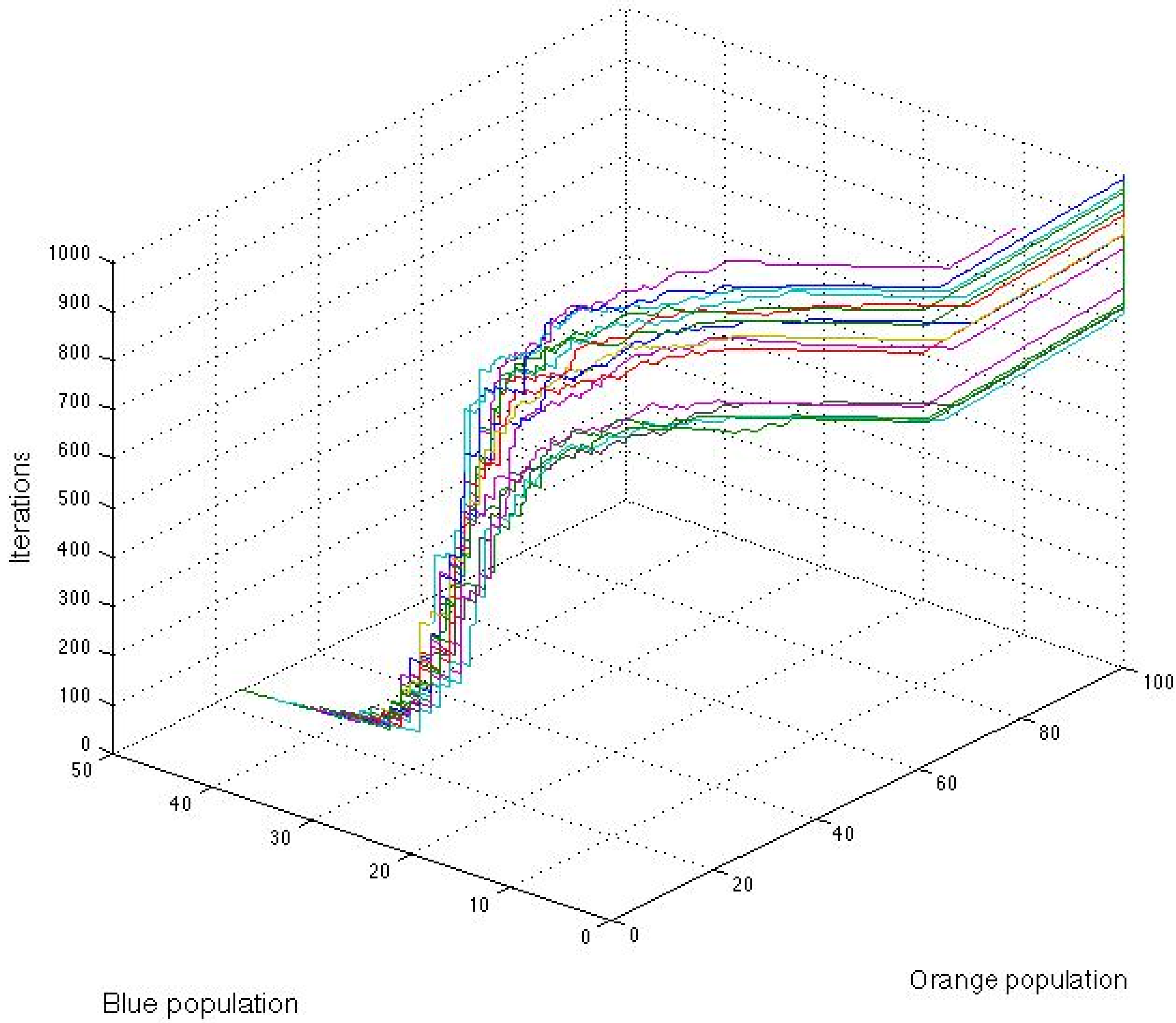


Figure 7.9: Three dimensional flow of different seeds, from initial condition $(25O, 50B)$. Different runs are initialised at $(25O, 50B)$, the trajectory of the system is plotted onto a three dimensional graph. When the population reaches approximately $(25O, 35B)$ the flow slows enough to maintain a mixed population for up to 700 iterations. However, the instability of the state, and the fact that only the B population leaves, means all the populations eventually segregate $(M = 0)$.

affected have populations less than the sample size (*i.e.*, they revert to the Schelling model). By $S = 50$ the change is noticeable for initial conditions with a population above 50. When $S = 25$ the results are most similar to the random movement change, with the divide blurring along $R_{BO} \approx 2 : 1$. By $S = 10$, whilst the divide is broadly the same shape, the region of uncertainty has increased. The dynamic result provided evidence that reducing S increased I, with a significant increase (median of $I = 50$) when $S = 10$. Further exploration has shown the increase is due to the system sustaining mixed populations for hundreds of iterations (see Figure 7.9). Whilst, this is a promising result, only the B population would only leave. Once they had left, they would not return and the O population begins to dominate the neighbourhood. These results offer evidence that H_0 can be rejected. Although mixed populations arose, and were briefly maintained, the populations would all eventually segregate. The results from the experiment show that reducing the knowledge of the agents increases the number of iterations taken to reach a steady state. Indeed, for $S = 10$ unstable mixed populations appeared; however, eventually, all the populations segregated. This highlights the precariousness of mixed populations, compared to the segregated populations. Once segregated the sampling has no effect, since no matter what the sample will always be the same. When the population is mixed the ratio of the sample can change, displacing the balance. This is something that could be explored further as, it would appear, the system is beginning to sustain mixed populations. It is possible that the slowing of the population dynamics, could be a route towards increasing the stability of mixed populations. If agents were prevented from leaving the moment $\tau < R$, it is possible that the flow of other agents could alter R enough such that $\tau \geq R$.

Chapter 8

Should I stay, or should I go?

> "If a person's limit is exceeded in this area he will go someplace else–a place, presumably, where his own color predominates or where color does not matter.." [74, p167]

8.1 Introduction

In Schelling's model, the moment an agent can no longer tolerate the ratio of the neighbourhood, they leave. Previous results (Section 6.6) have shown that, although changing the order of movement can slow the population dynamics, the model is unable to produce anything other than complete segregation. In the Bounded Neighbourhood Model, agents in the population move as soon as they are 'sad', to any place that is more tolerable. However, movement should have a cost. If agent movement was restricted, the system might change enough for the agent to remain.

In this modification, we define a 'sadness' parameter. Now, the movement of the agents is based on a probability derived from their level of 'sadness' relative to the second neighbourhood. Agents attempt to minimise their 'sadness' by switching to the other neighbourhood. By considering the problem in terms of energy minimisation, a friction parameter F is defined to indirectly 'control' the flow of the system. These characteristics are added to the Domain model in Section 8.3, and from there (following the CoSMoS process) the Platform Model is adjusted. The Simulaton Platform is used to run the model and produce results that are used to test the hypothesis described below.

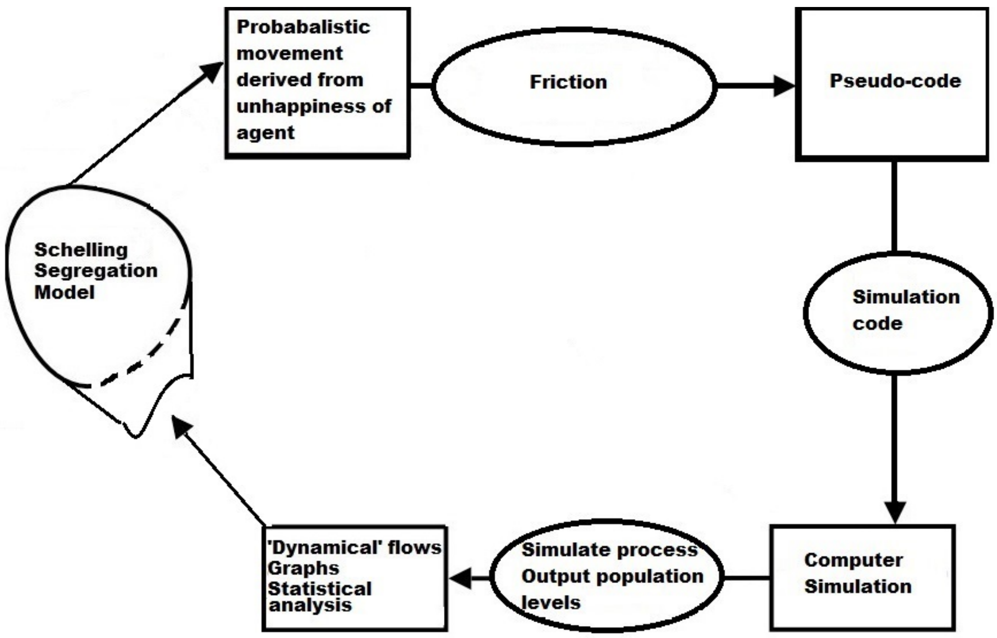


Figure 8.1: An application of the CoSMoS process in testing the effect of agents' instantaneous movement. A 'friction' parameter (F) is added to the domain model. The parameter can be adjusted to explore the effects, with systematic testing constant with previous experiments.

8.2 Hypothesis

The null hypothesis (H_0) argues that, increasing F will have no effect on the results of the model. The static and dynamic measures (M and I) are used in the analysis of results, and testing of the hypothesis.

8.2.1 Static measure

The static measure analyses the final population configurations. Using Equation (4.1) the mix of the population M is measured and compared to the results from Schelling's model, where $M = 0$. Values of $M > 0$ indicates mixed populations have arisen.

8.2.2 Dynamic measure

The dynamic measure analyses the flows of the populations into and out of the neighbourhood. The U test (described in Section 4.4.3 is used to analyse the results of the number I of iterations, testing the effects of the changes of the model. Results that return significant results undergo further analysis using the A statistic (described in Section 4.4.5).

8.3 Domain model: adding friction

The parameters and language of the Domain Model are retained (see Section 5.1). The 'sadness' of an agent ($o_n \in O$)[1] is calculated thus

$$o_n(u) = R_{BO} - o_n(\tau) \tag{8.1}$$

where $o_n(\tau)$ is the tolerance of the agent o_n and R_{BO} is the ratio inside the neighbourhood (see (5.3)). However, problems arise with agents who find themselves "outside" the neighbourhood ($o_n \in (O \cap E^*)$), but are calculating their 'sadness' based on the ratio inside (R_{BO}, see equation (5.5)). Agents that are inside ($o_n \in (O \cap E)$) now have different behaviours from those outside (*i.e.*, if an agent is 'sad' inside they move but if they are 'sad' outside they stay outside).

8.3.1 Changes to the model

In the extension considered in this chapter, agents now utilise the ratio of a second neighbourhood E^*. Introduced in Section 5.1, E^* was simply (in Schelling's words) "someplace

[1]As with the previous Domain Model the agent $o_n \in O$ has an equivalent agent $b_n \in B$.

else" [74, p168]. In this extension, E^* is defined as the neighbourhood where the agent is considering moving to. Thus, the ratio of E^* is needed to allow a comparison and (for the O population) is given by the ratio

$$R_{BO^*} = \frac{(B \cap E^*)}{(O \cap E^*)}. \tag{8.2}$$

The agents o_n use this ratio in the equation

$$o_n(u^*) = R_{BO^*} - o_n(\tau), \tag{8.3}$$

to calculate their 'sadness' if they switch neighbourhoods. By comparing this value with u, the 'sadness' difference Δu is calculated,

$$\Delta u = u - u^*. \tag{8.4}$$

The same equations are used to calculate the Δu of the B population. It is reasonable to assume that smaller Δu is, the more likely an agent will move. This can be modelled as follows. Substitute Δu for x into the exponential function e^{-x} and dividing the exponent $-\Delta u$ by a friction parameter (F) to obtain a function $e^{-\Delta u/F}$ that decreases steeply to 0 as Δu decreases and increases up to 1 as F increases. The movement of agents, can be modelled by a probability derived from Δu and 'controlled' by F:

$$P(\text{stay}) = \min\{1, e^{-\Delta u/F}\}. \tag{8.5}$$

This means that, as Δu increases, the probability of staying decreases; as F increases the probability of staying increases (Figure 8.2). Thus, instead of agents moving as soon as they are sad, their movement is controlled by a probability, based on the difference between the ratio of populations. If $\Delta u < 0$, agents will not move, since it implies that $R^* > R$, where R and R^* are the respective neighbourhood ratios.

8.4 Platform Model

Using the structure and language developed in Section 5.2, the Platform Model is presented using algorithms 12 and 13. Agent movement is based on the calculation of $P[stay]$, shown in the movement calculation algorithm 13. Now the question remains as to the order of movement. In Schelling's model, agents moved "in order of their tolerance" [74, p167], if agents are moving into, and out of, two neighbourhoods, both of which are now being

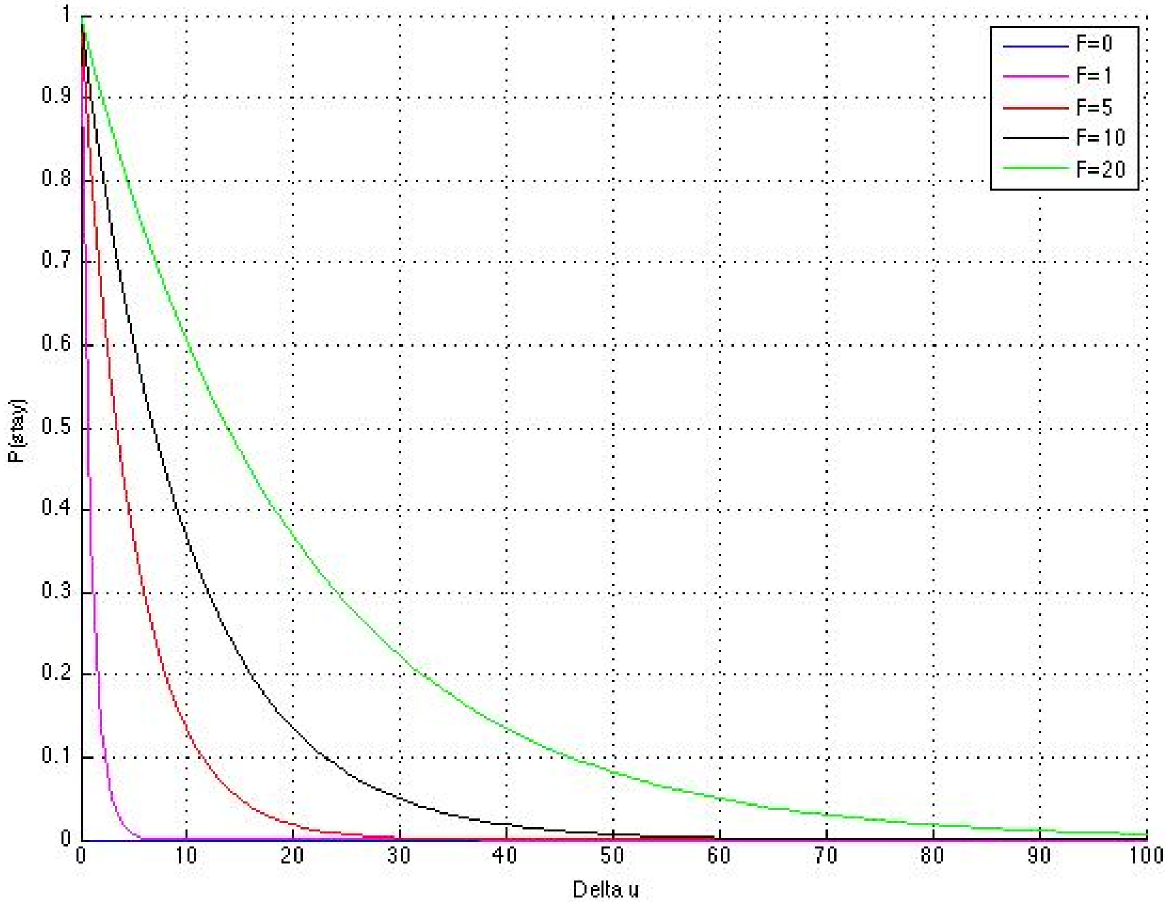


Figure 8.2: Probability distributions derived from eq (8.5) with $F = 0, 1, 5, 10$ and 20.

Algorithm 12 Probabalistic Bounded Neighbourhood Model $\mathcal{I}$ Initialisation/Iteration Operator

Create population of agents $O \cup B$
apply τ to agents
set $O \cap E$ and $B \cap E$ agents state = [in]
while $o_n(\tau) < R_{BO}$ or $b_n(\tau) < R_{OB}$ **do**
    execute Algorithm 13
end while

Algorithm 13 Agent type o movement calculation: one timestep

for each agent $o \in O$ **do**
    set Δu
    if selected? $=$ *false* **then**
        set selected? *true*
        if $e^{-\Delta u/F} < URD[0,1]$ **then**
            move o_n
        end if
    end if
    if $\nexists o_n$ with selected? $=$ *false* **then**
        set agents $o \in O$ selected? $=$ *false*
    end if
end for

considered, the agent with the minimum τ will be the first to have the opportunity to move (least tolerant moves first). If their 'sadness' is not strictly decreased, the agents will not move but will be immediately selected again. This can be overcome by adding a selection flag to algorithm 13, which will only allow re-selection of an agent once all other agents have been selected. Note that whilst not all agents selected will move, those that choose not to move are still flagged. This technique means (for the deterministic model $F = 0$) I will have a minimum value of 150 (the number of agents in the model).

8.5 Simulation Platform

Netlogo is used to implement the modified segregation model. The platform is discussed in Section 5.3 and is used for two main reasons. Firstly, it is important to keep the process as consistent as possible. Since there is no reason to change the Simulation Platform, it is sensible to retain it. Secondly, Netlogo has a number of useful tools that facilitate experimentation. One of these (`BehaviorSpace`) is able to run a number of simulations across a range of parameters. Whilst this is an extremely useful tool, there is a major drawback. The output of (`BehaviorSpace`) is cached in Netlogo until the entire parameter space has been explored. This means that as the number of runs increases the system slows exponentially. Whilst a solution to this problem would be an interesting aside, time pressure dictates capping the maximum number of iterations to 1500.

8.6 Results Model

The simulation is used to test the hypothesis from Section 4.4. The null hypothesis (H_0) is that changing F will have no effect on the outcome of the simulation. The parameter F is run at five values: 0 (for validation purposes), $1, 5, 10$ and 20. The probabilistic distributions of these values are shown in Figure 8.2. These values, whilst arbitrary in some aspects, nonetheless, offer a good sample representation of a range of possible values. It is believed that, as F increases, the flow will decrease and, possibly, create new stable mixed populations. The simulation is tested over all possible initial population conditions, from (o_1, b_1) to (o_{100}, b_{50}), returning 5000 samples. In addition each initial condition is run 25 times. Whilst this dampens the possible effects from using a stochastic process, such a large sample size means a significant result is probable. To check for an effect size on significant results, the Vargha-Delaney test is applied. As stated earlier, because of memory constraints, the maximum number of iterations is capped at 1500.

8.6.1 Validation

The validation of the model is achieved by setting $F = 0$, thus (from equation (8.5)) $P(\text{stay}) = 0$ when an agent is 'sad'. With this setting the movement parameter of the model reverts to a deterministic model. Whilst it was hoped this would replicate Schelling's Bounded Neighbourhood Model, the addition of a second neighbourhood has an effect on the results. In this adaption, agents switch neighbourhoods only if $R^* > R$. The symmetry of the neighbourhoods means that the population flows diverge along the $2 : 1$ ratio (Figure 8.3). If an initialisation has both neighbourhoods with an equal ratio, the system freezes. This is shown by dots, representing the unchanging populations flows, along the diagonal in Figure 8.4. This causes the results of the simulation to return $M = 0.005$ compared to Schelling's $M = 0$. However, this result is an artefact of the populations that initialise with $\Delta u = 0$, meaning no agents will move. In addition, the flows of the population indicate that the entire majority population has to be present before the minority population begin to leave. This causes the median value of I to jump from 41 to 149. The results from the validation show that the flows do not replicate those of Schelling, suggesting a significant change to the model. Whilst

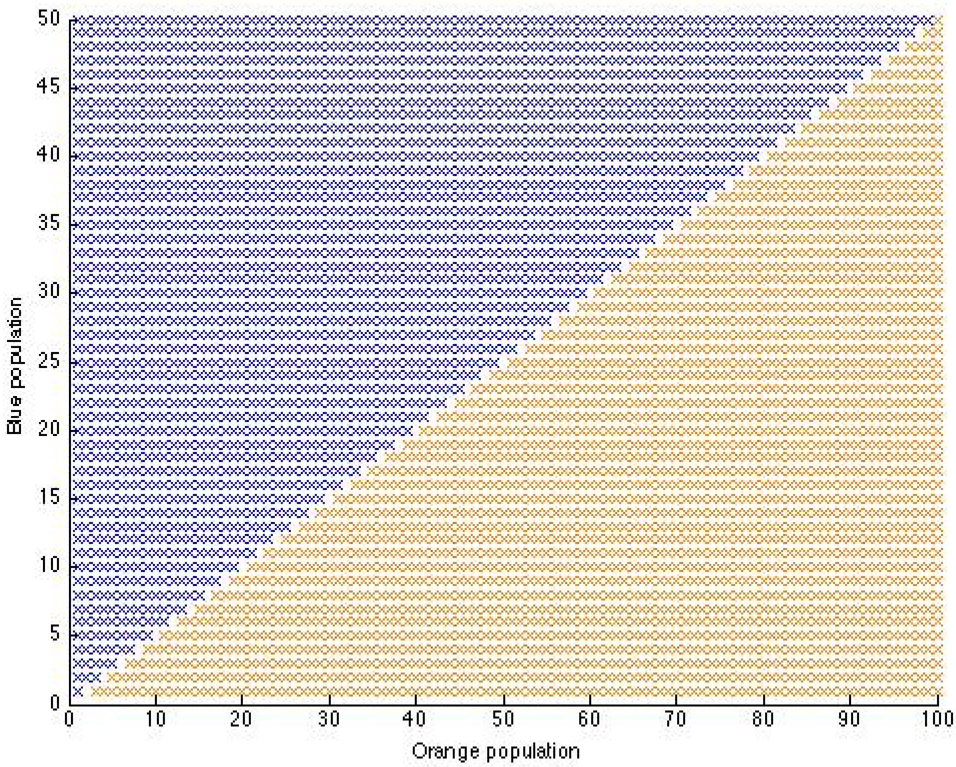


Figure 8.3: A colour map of the adapted segregation model with $F = 0$. The addition of a second neighbourhood means the populations clearly divide across the 2 : 1 ratio. Blue crosses represent final configurations of the entire B population and none of the O. Whilst orange crosses represents final configurations of all O and no B. Populations that initialise on the boundary have symmetrical neighbourhoods. This means $\Delta u = 0$ and no agents will move.

this does mean it is difficult to compare the results to the other experiments presented, it does not render the experiment meaningless.

8.6.2 Static analysis

The results of the static measures can be seen in Figures 8.5 and 8.6, and are displayed in Table 8.1. Although M returns non zero values for all values of F, the majority of these results are populations initialising with equal neighbourhoods. This initialisation causes $\Delta u = 0$ and means no agents will move, in effect the system is frozen. Knowledge of this artefact means we can discount values of $M \leq 0.005$. Although initial results returned values of $M > 0.005$ for $F > 5$, these were just artefacts from the capping of I. Increasing the value of I leads

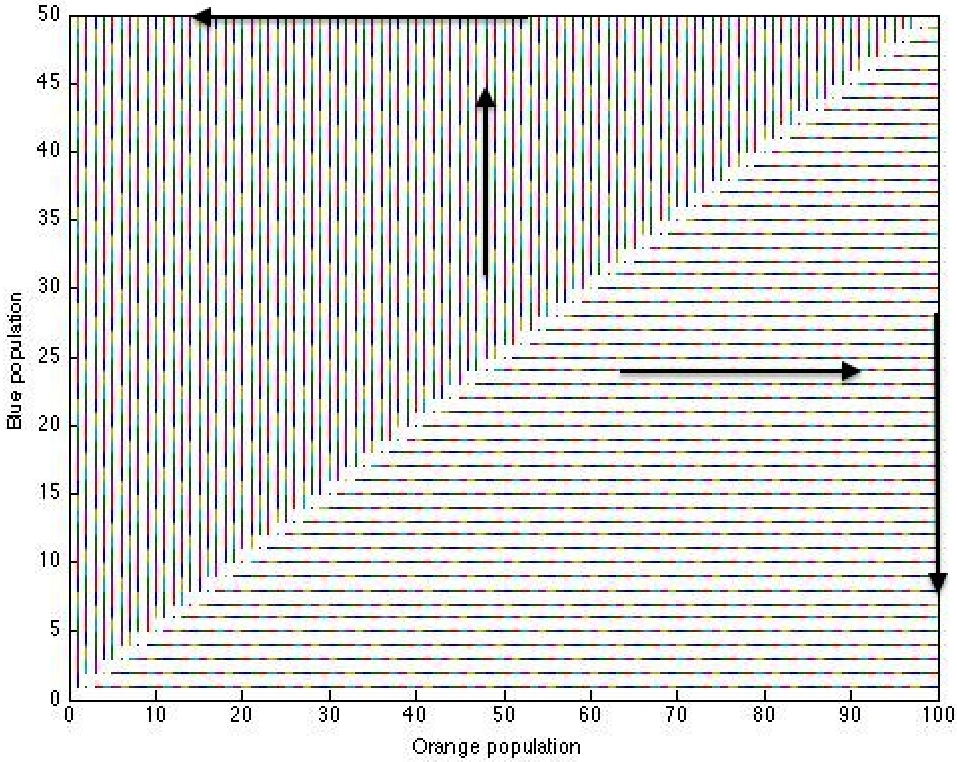


Figure 8.4: The population flows of the adapted segregation model. The addition of a second neighbourhood means the populations clearly divide across the 2 : 1 ratio. Note the unchanging populations along the divide, seen as single dots. These represent populations with $\Delta u = 0$, meaning no agent will move ($P(\text{stay}) = 1$). Since for each of these populations $M > 0$, the artefact is carried through to the results causing $M = 0.005$.

F	Median I	p	A_{12}	M
0	149	1	-	0.005
1	220	0	$> \mathbf{0.71}$	0.005
5	588	0	$> \mathbf{0.71}$	0.005
10	1088	0	$> \mathbf{0.71}$	0.005
20	2214	0	$> \mathbf{0.71}$	0.005

Table 8.1: Results from different F values. The dynamic results show a significant result, with a large effect size for even the smallest value of F. Although the results for M indicate mixed populations they are actually artefacts of the simulation. The M value is a result of the populations that initialised with perfectly balanced neighbourhoods (*i.e.*, $\Delta u = 0$), meaning no agents will move.

to $M = 0.005$ (see Table 8.1). As figures 8.7 and 8.8 show, the populations are still clearly segregated along the $2 : 1$ boundary.

8.6.3 Dynamic analysis

The results from the dynamic analysis are displayd in Figures 8.7 and 8.8. They show that as F increases, the flow of the population is disrupted (i.e., the number of iterations I increases). This can be seen in a comparison of the population runtimes shown in Figure 8.9. When $F = 0$ the median $I = 149$, setting $F = 1$ increases the median $I = 220$. This increase is due to the system having to iterate through the population twice more, before reaching stability. This can be seen by the plateaus in the distributions every 150 iterations. As F increases further, increase between plateaus is reduced, suggesting less systems stabilising at each round of iterations. It is noticeable that when $F > 5$ the maximum I is reached before the system reaches stability. Although for $F = 5$ this only happens in 5% of cases, for $F = 10$ it has increased to 35% and, by $F = 20$ around 66% of populations have failed to reach stability. This artefact means the median I for $F = 20$ is 1500. By increasing the maximum I the results for $F = 20$ return a median $I = 2214$. However, adjusting F had no effect on populations initialised with identical neighbourhoods. This is because of $\Delta u = 0$ so, no matter the value of F, $P(stay) = 1$.

8.7 Conclusions

The work presented in this chapter has explored the effect of restricting agent movement. This has been tested by introducing a parameter F to control the ease of movement in the

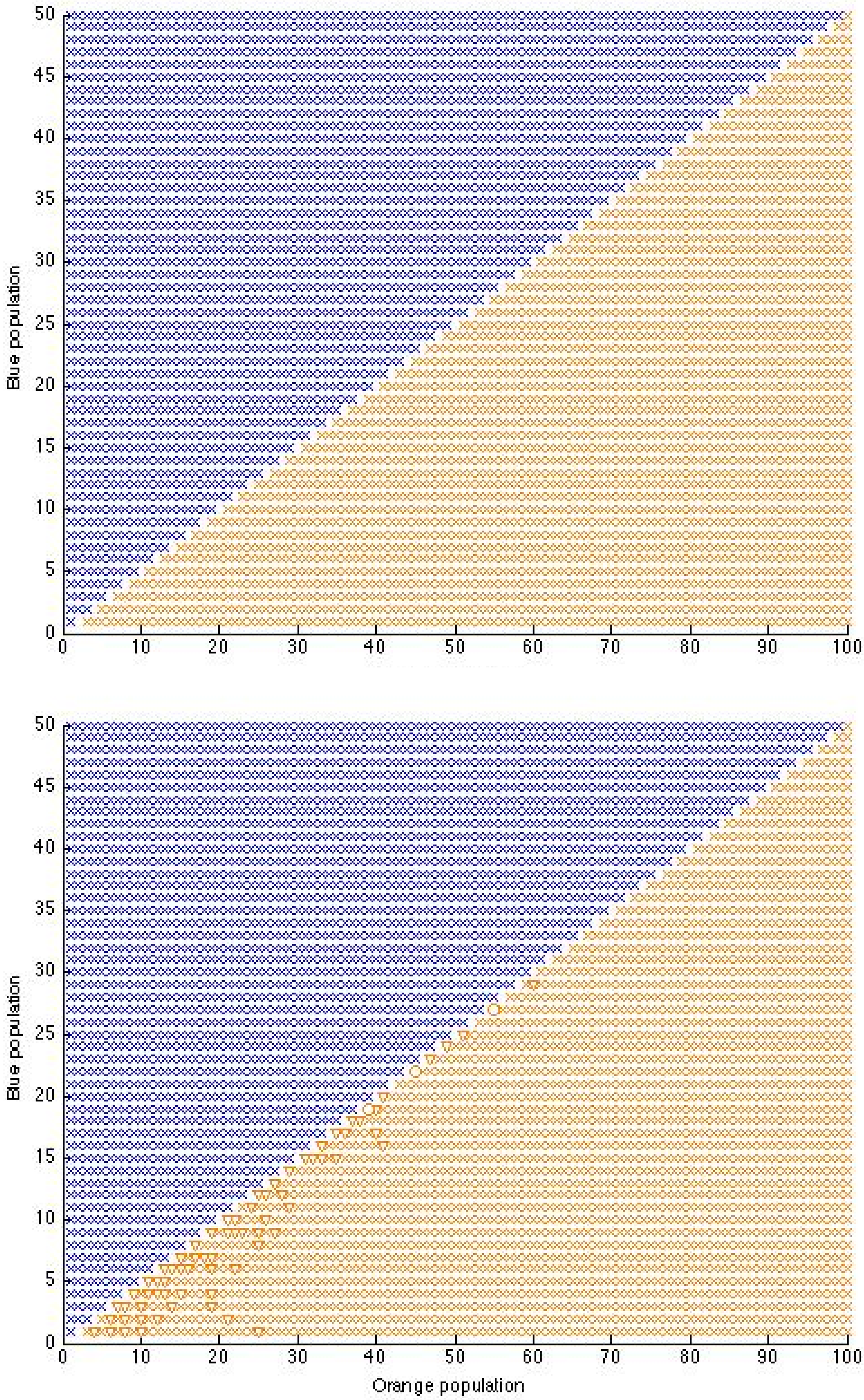
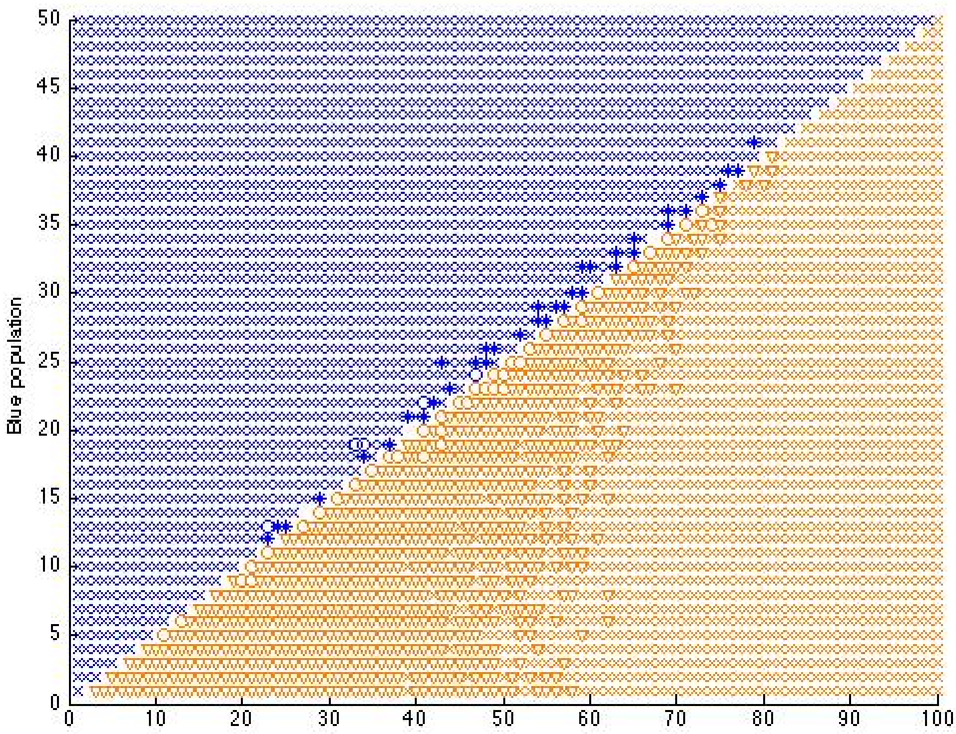


Figure 8.5: Colour map of the final population configurations for $F = 1$ (top) and $F = 5$ (bottom). As previously, the location on the map relates to the initial condition of that final population configuration. When $F = 1$ the results are identical to $F = 0$. When $F = 5$, some populations have failed to reach a steady state. Here orange triangles are populations that contain no Bs, but not all of the Os have entered. Whist orange circles are populations with $R_{OB} > 4$.



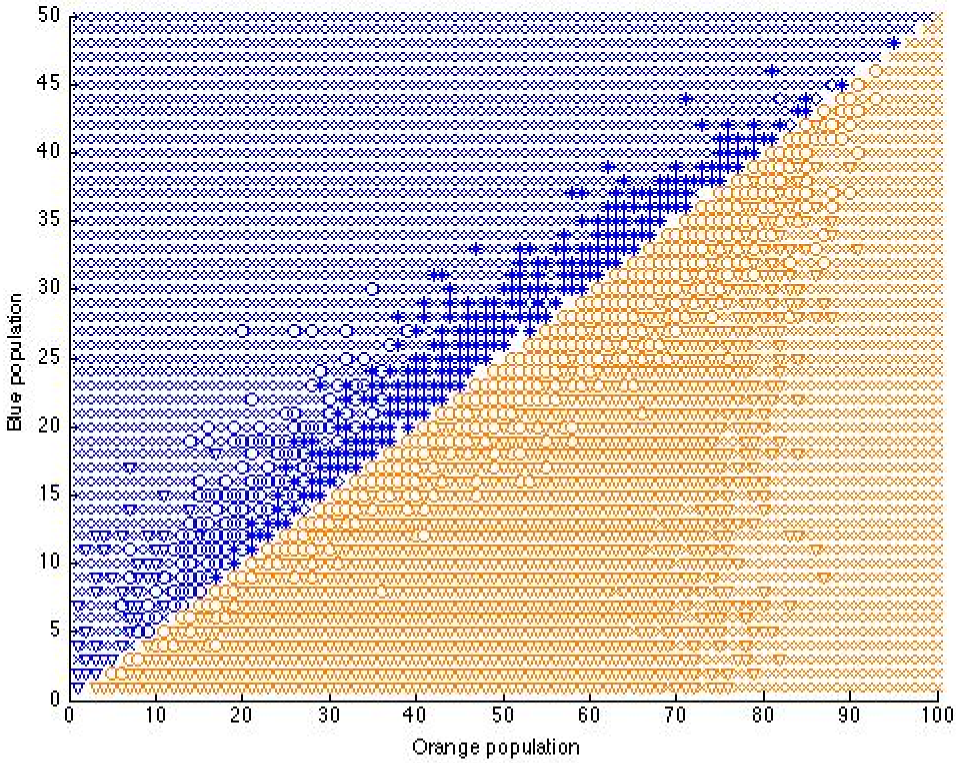


Figure 8.6: Colour map of the final population levels. Increasing $F = 10$ (top) and $F = 20$ (bottom) produces even more populations failing to completely segregate. However, once again, this is due to the limit of I. By increasing the limit, the populations eventually segregate. As with the previous mapping, crosses indicate fully segregated populations, triangles indicate an absence of the 'other' population whilst circles show populations $R > 4$ but still contain a mixture of agents. In addition blue stars indicate populations with $R_{BO} > 3$.

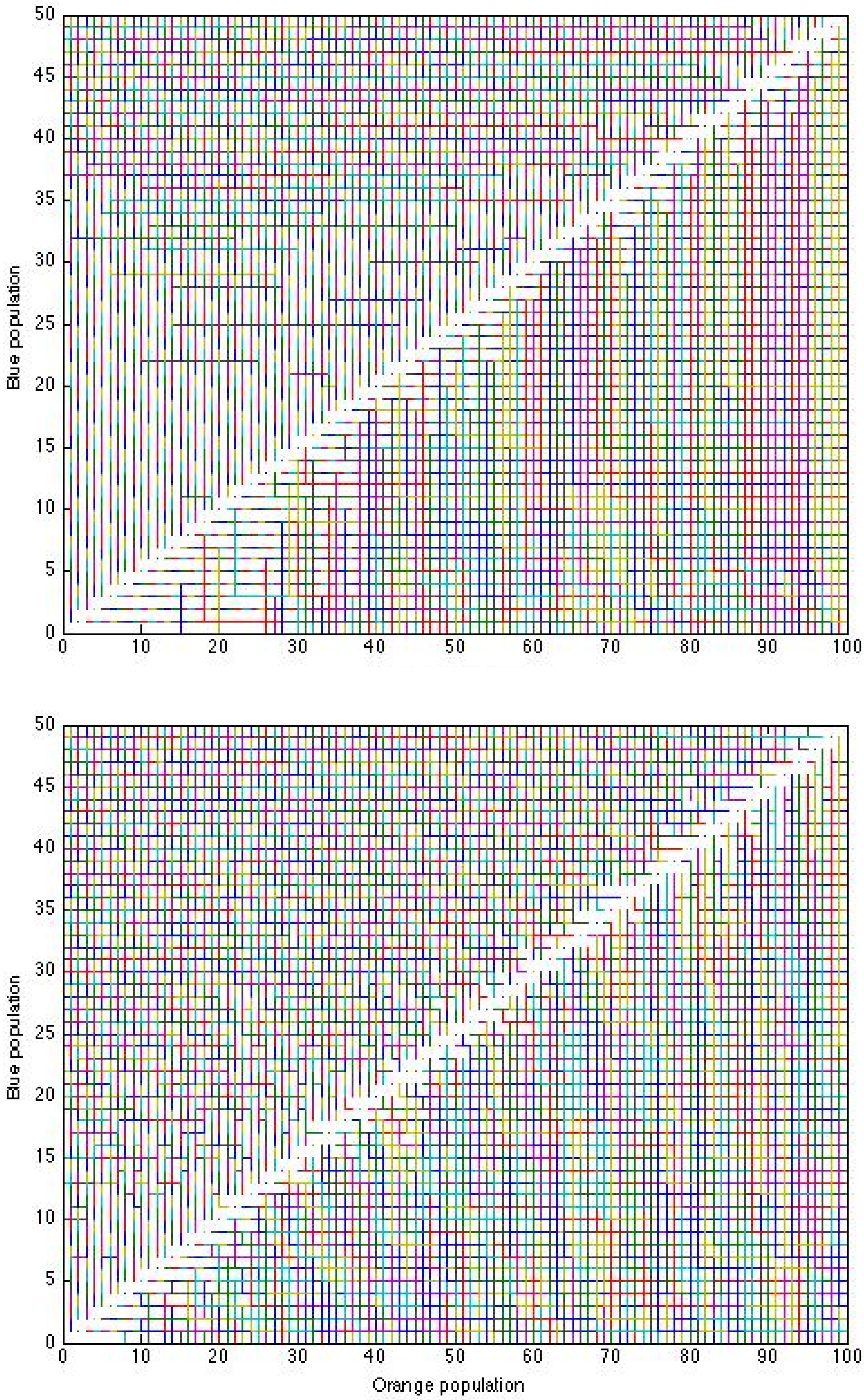
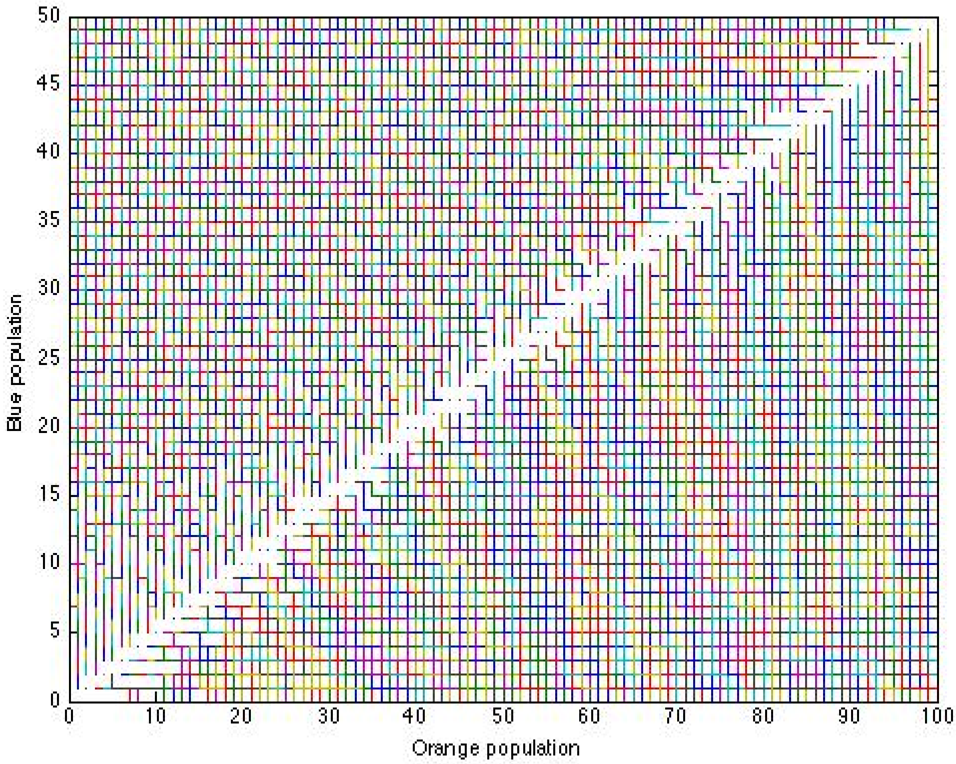


Figure 8.7: The population flows of the adapted segregation model with $F = 1$ (top) and $F = 5$ (bottom). The unchanging populations along the 2 : 1 ratio diagonal are consistent with the $F = 0$ results and are unaffected by varying values of F. The increase in F disrupts the flows of the populations, but is unable to maintain stable mixed populations.



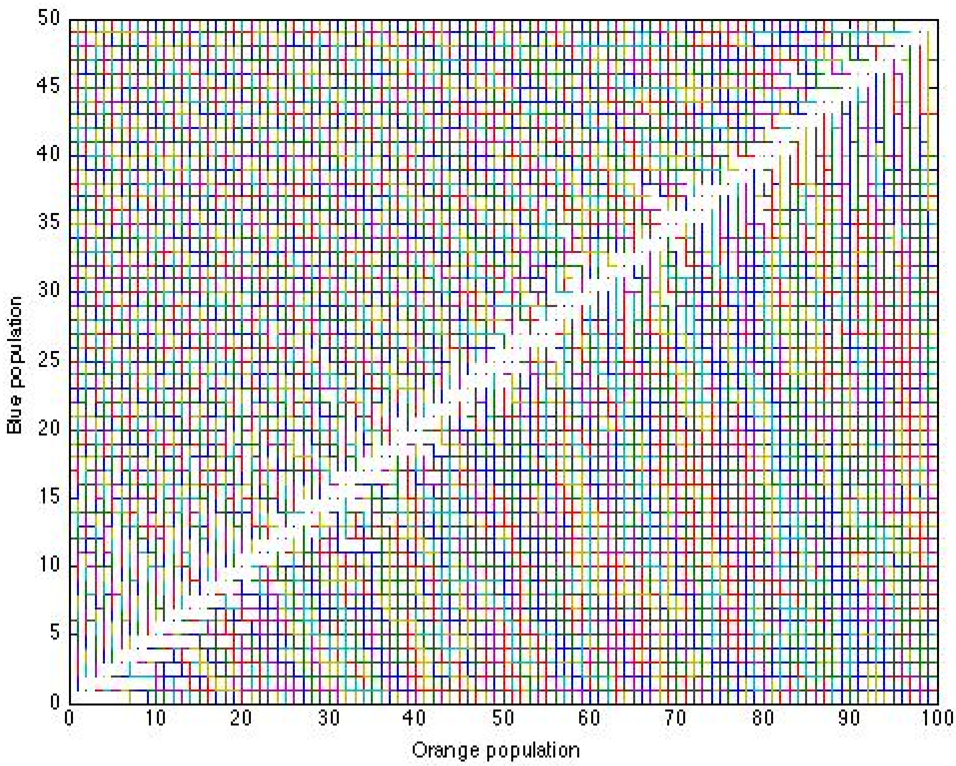


Figure 8.8: The population flows of the adapted segregation model with $F = 10$ (top) and $F = 20$ (bottom). The increase in F further disrupts the flows of the populations, but is still unable to create any stable mixed populations.

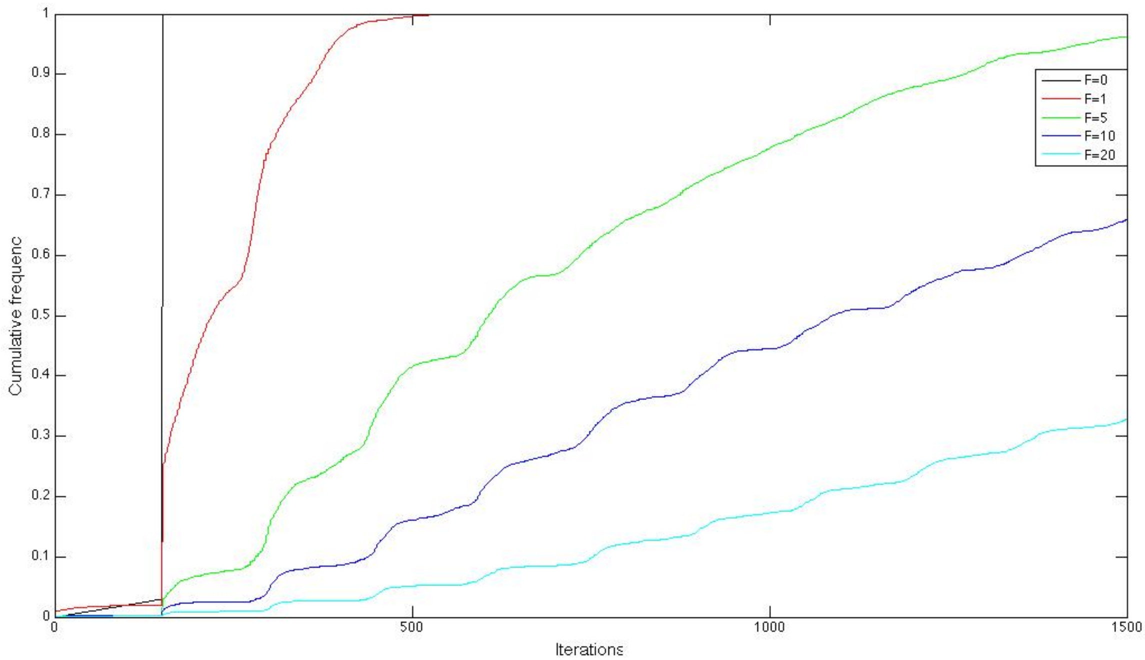


Figure 8.9: Distribution of population runtimes for the different values of F. Because of the selection process, when $F = 0$, I is always the number of agents in the population (here 150). This is shown by the straight vertical line at 150 iterations. When $F = 1$ the distribution clearly plateaus at 300 and 600, with less obvious plateaus at 150 and 450. These plateaus indicate the system cycling through the population of agents. As F increases further, increase between plateaus is reduced, suggesting less systems stabilising at each round of iterations. It is noticeable that when $F > 5$ the maximum I is reached before the system reaches stability. Although for $F = 5$ this only happens in 5% of cases, for $F = 10$ it has increased to 35% and, by $F = 20$ around 66% of populations have failed to reach stability.

model. The adjustment has meant a second 'comparison' neighbourhood has been defined. The validation of the model has provided evidence that the model produces different results in both static and dynamic analysis. However the static results returning $M > 0$ have been shown to be artefacts created from the calculation of Δu. The failure of the validation means the model cannot be considered a valid replication of Schelling's Bounded Neighbourhood Model. This makes the results difficult to compare to either Schelling's model, or the results of other experiments. However, the results of the experiment are interesting, since they show that even heavily restricting the movement of the agents is unable to prevent total segregation. The experimentation on varying values of F, did initially suggest that the model maintained mixed populations ($M > 0$). However, further analysis highlights two artefacts causing the result. Firstly, when the system initialises with the two neighbourhoods having equal ratios, no agents move. This is shown by the single dots, signifying unchanging populations, along the diagonals of Figures 8.4, 8.7 and 8.8. The agents in these populations will not move, because the identical ratio means $\Delta u = 0$, leading to $P(stay) = 1$ and $M = 0.005$. Secondly, as F is increased I also increases; however, since the maximum of $I = 1500$, the remaining mixed populations are merely in the process of segregating. By increasing the maximum value of I, the second artefact disappears. The results indicate that, the change in the model produces a large effect on the dynamic results but, even though stable mixed populations are maintained, they are shown to be just an artefact of the model. The increase in I is a result that is to be expected, since the adjustment to the model is the first time the movement of the agents has been restricted. However, the failure of the validation test makes a comparison between these results, and those from Chapters 6 and 7 difficult. This means that, although the change in the model shows a significant increase in I, it cannot be known if the results are due to the change in the movement parameter, or the change in the environment. The adjustment means that, although the model is able to replicate Schelling's segregation, it is unable to recreate Schelling's flows. This reduces its defendability as a model of Schelling's Bounded Neighbourhood. However, the fact that a second neighbourhood can cause such an effect on the model is an important result. This suggests that the environment 'outside' the model needs much more careful consideration before being applied, something that is often disregarded in social interaction models.

Chapter 9

Friends and Neighbours

9.1 Introduction

Networks have been widely used to model social systems, such as group membership [42] and, interestingly, segregation [38]. Networks are described in Section 2.4, so here we provide a brief introduction to the additional concepts and terms needed for this chapter. Networks are described in terms of nodes and their connections to other nodes. They are often characterised in terms of distances between nodes (path length), connectivity (degree) and density ρ. It is natural to interpret nodes connected to a network being agents inside a neighbourhood, and vice-versa. By describing Schelling's Bounded Neighbourhood Model as a network, we can exploit network techniques. As with the previous Bounded Neighbourhood Model, the ratio of the neighbourhood network changes over the time of the run. As the ratio changes, nodes (which, for consistency, we call agents) connect and disconnect from the neighbourhood, until no more change their connection. Implementing a social network creates a subset of agents in each of the population types. Those inside the social network alter the calculation of the neighbourhood network. This allows the investigation of the effect, if any, of the differentiation by type alone (*i.e.*, there is no difference between agents other than their type).

9.2 Hypothesis

The hypotheses tests the effects of recasting the model. We offer the null hypothesis (H_0) that recasting the model will have no effect on the outcome of the simulation. This is tested using two measures, static and dynamic (discussed in Section 4.4).

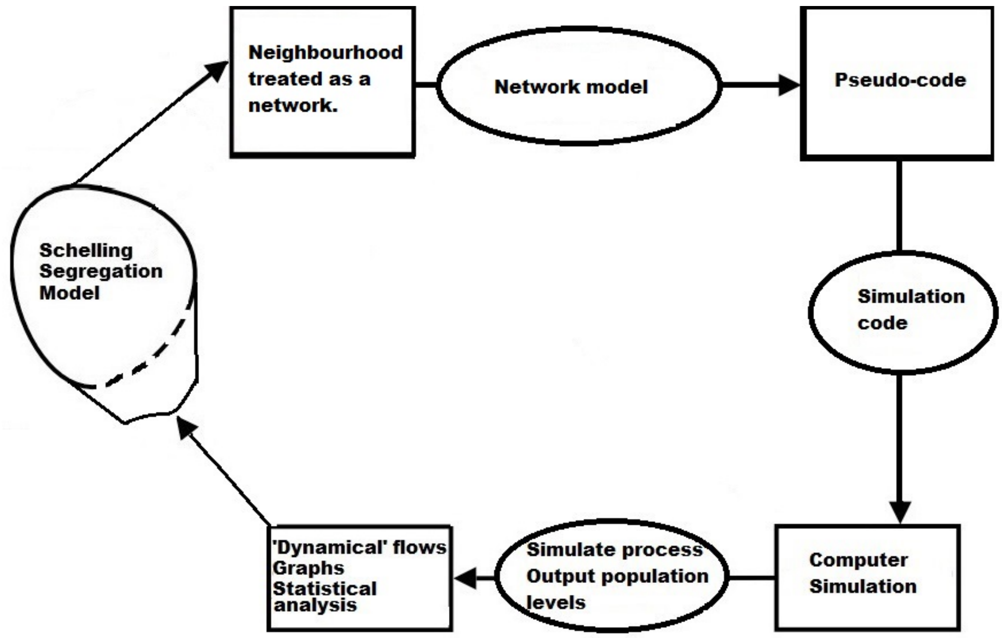


Figure 9.1: Using the CoSMoS procecss to build a network model and simulation of Schelling's Bounded Neighbourhood Model.

9.2.1 Static measure

The static measure of the final population inside the neighbourhood, shows whether the alteration to the model produces stable mixed populations. Following equation (4.1), M is used to measure the final population configurations. The hypothesis, that changing the movement rule will produce mixed populations, is tested by comparing M to Schelling's findings of $M = 0$. A result of $M > 0$ suggests mixed populations have arisen, counter to Schelling's model.

9.2.2 Dynamic measure

The analysis of the population flows focuses on I, which is the number of iterations taken to reach a steady state. The system has reached a steady state when no more agents wish to move. The results are analysed using the Mann-Whitney-Wilcoxon U test and the Vargha-Delaney A-statistic, described in Section 4.4.

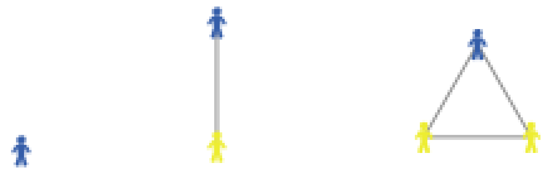


Figure 9.2: Translating Schelling's Bounded Neighbourhood model as a network. Agents are nodes. At left, an agent forms a single node; in the centre, two agents inside the neighbourhood are connected (an edge). On the right, three connected agents form a triangle.

9.3 Domain model: Schelling's Bounded Neighbourhood Model as a network

Following the formalisation presented in Chapter 5, the Domain Model is constructed as a network model and, from there, a Platform Model is built and simulated. To validate the simulation, results are measured against the results from the original Bounded Neighbourhood Model simulation. The validation is followed by an experiment, testing the assumption of binary differentiation by type. This experiment implements a social network between the nodes.

A simple translation of the model into a network is illustrated in Figure 9.2. The network consists of $n = 150$ agents, with 100 orange ($o \in O$) and 50 blue ($b \in B$). Agents are considered as nodes with two variables: a type (here o or b) and a tolerance τ, drawn from the distribution shown in Figure 3.5. Inside the neighbourhood component, $n' \leq n$ agents are connected by m links, with the distribution

$$p_k = \begin{cases} 1 & \text{for all agents inside the neighbourhood} \\ 0 & \text{for all agents not inside the neighbourhood.} \end{cases} \tag{9.1}$$

The adjacency matrix

Following Section 2.4 an $n \times n$ adjacency matrix is constructed from

$$A_{ij} = \begin{cases} 1 & \text{if there is a link between nodes } i \text{ and } j \\ 0 & \text{otherwise;} \end{cases} \tag{9.2}$$

where i and j are i and jth agents in the network. This gives the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & A_{1,2} & \dots & A_{1,149} & A_{1,150} \\ A_{2,1} & 0 & \dots & A_{2,149} & A_{2,150} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{149,1} & A_{149,2} & \dots & 0 & A_{149,150} \\ A_{150,1} & A_{150,2} & \dots & A_{150,149} & 0 \end{pmatrix} \tag{9.3}$$

Agents connected to the neighbourhood are defined as in E, and are fully connected to all other agents in E. If $|E| \equiv n'$, where $|E|$ is the cardinality of E, then the number of connections m can be calculated thus,

$$m = \frac{n'(n'-1)}{2}. \tag{9.4}$$

The ratio for the O population, R_{BO}, is derived from equation (5.3) giving:

$$R_{BO} = \frac{n'(B)}{n'(O)}. \tag{9.5}$$

Similarly for the B population,

$$R_{OB} = \frac{n'(O)}{n'(B)}. \tag{9.6}$$

Agents that are connected with $\tau < R$, disconnect from the network. They disconnect from smallest to largest, in order of τ and one at a time. When an agent disconnects from the network (i.e., removes all links) they become completely disconnected, and become a node (or component of one node). Amongst these components, if R changes enough such that $R \geq \tau$, the agent with largest τ reconnects. Thus, the network is made up of one major component (the neighbourhood), where all nodes are fully connected, whilst the rest are components of single nodes. The number of components in the network can be calculated by $(n - n') + 1$; and can be used to verify the density of the network.[1]

9.4 Platform Model

The dynamics of the model are formalised in algorithms 14 and 15. The advantages of this approach are discussed in Chapter 5. Briefly, the pseudo-code description is a useful stepping stone from the model to the simulation. It gives an accurate representation of the model that is easily translated into code.

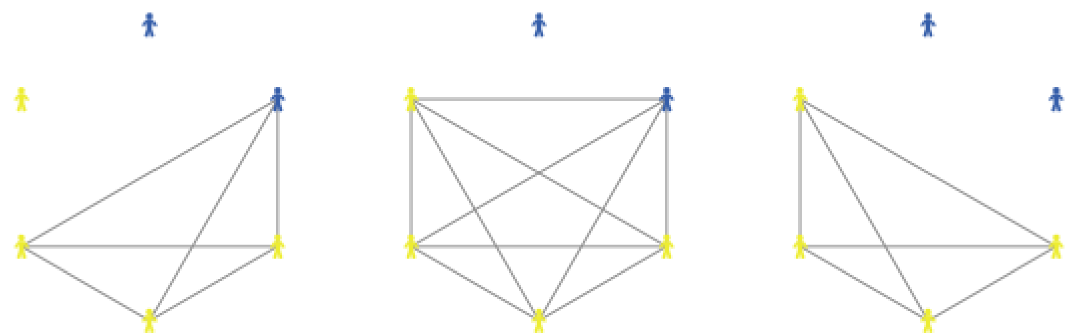
[1]Since, as $n' \to n$, the number of components $\to 1$.

Algorithm 14 Bounded Neighbourhood Network Model Initialisation

```
create population of n = 150 agents
apply type to agents
apply τ to agents
fully connect n′ ≤ n agents
while any n ∈ n′ ∧ n [sad] or n ∉ n′ ∧ n [happy] do
  execute algorithm 15
end while
```

Algorithm 15 update: one timestep

```
calculate neighbourhood ratio R {see equation (5.5)}
for each node n do
  if R ≤ n(τ) then
    set n [happy]
  else
    set n [sad]
  end if
end for
if ∃n ∉ n′ ∧ n [happy] then
  fully connect n with max τ to network → n ∈ n′ {(a) happiest outside connects}
else
  if ∃n ∈ n′ ∧ n [sad] then
    fully disconnect n with min τ → n ∉ n′ {(b) unhappiest inside disconnects}
  end if
end if
```



$$\mathbf{A_1} = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&1&1&1&0\\ 0&1&0&1&1&0\\ 0&1&1&0&1&0\\ 0&1&1&1&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \quad \mathbf{A_2} = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&1&1&1&1\\ 0&1&0&1&1&1\\ 0&1&1&0&1&1\\ 0&1&1&1&0&1\\ 0&1&1&1&1&0 \end{pmatrix} \quad \mathbf{A_3} = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&1&1\\ 0&0&1&0&1&1\\ 0&0&1&1&0&1\\ 0&0&1&1&1&0 \end{pmatrix}$$

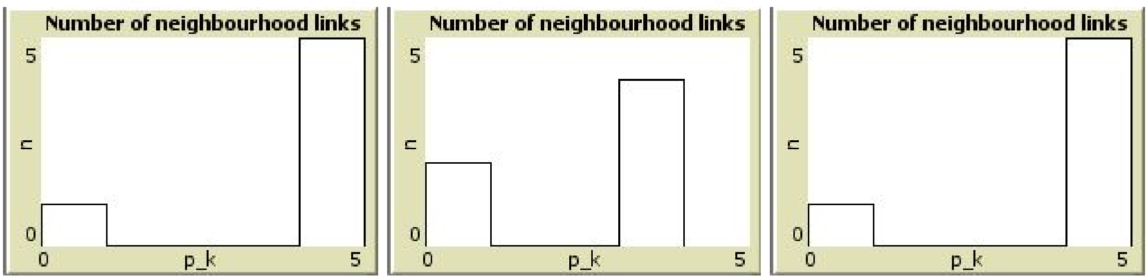


Figure 9.3: A simple run of the Platform Model, showing the resulting networks (top), adjacency matrices (middle) and distributions of p_k (bottom).

9.5 Simulation Platform

The pseudo-code is translated into Netlogo. Figure 9.3 is the output of two iterations of algorithm 15. Nodes outside the network are completely disconnected (left). If R is below the τ of a disconnected node, they connect (centre). If R is above the τ of a connected node, they disconnect (right). The initial (left) and final (right) distributions are identical, whilst the intermediate network (centre) is connected to one more agent. The network initialises with $n = 6, n' = 4, m = 6$ and $\rho = 2/5$ (left). The zeros on the first and last rows of $\mathbf{A_1}$ correspond to the disconnected agents. Since the R of the network is acceptable for a disconnected agent (and since they have the max τ of their type) they connect. The network now has $n = 6, n' = 5, m = 10$ and $\rho = 2/3$. The addition leaves only one disconnected agent, highlighted in $\mathbf{A_2}$. In addition R has increased for the blue agents and decreased for the orange. Since no 'happy' agents exist outside the network, the removal of 'sad' agents is activated, leading to the final configuration (right). In this configuration, no agents change their connection to the network, and the run is terminated. In this case the network has an identical makeup to the first network with $n = 6, n' = 4, m = 6$ and $\rho = 2/5$. However the difference between $\mathbf{A_1}$ and $\mathbf{A_3}$, highlights the different nodes connected to the neighbourhood.

9.6 Results Model

To check the validity of the model, the results of the simulation are measured against the results from the original Schelling simulation (see Chapter 5). As with previous experiments, the same initial conditions are used (see Chapter 6). In network terms, each possible adjacency matrix is constructed. This gives 5000 matrices, which are run until a steady state is achieved. When the network reaches a steady state (defined as the point where no more nodes wish to connect or disconnect), the number of iterations is recorded (I) and the population configuration is measured (M). Since the model is deterministic, there is only need for a single run across all the initial conditions. As previously, due to memory constraints, the maximum value of I is capped to 1500.

The simulation results of the population dynamics, are measured against the results from the original Bounded Neighbourhood Model simulation (see Chapter 6). The comparison of flows (shown in Figure 9.4) suggests they are identical. However, to provide statistical evidence, the U test is applied. This is a non-parametric test that evaluates two different samples and estimates a probability that two samples are drawn from the same population. The use of the test is discussed in Section 4.4.3. The test is normally used to measure differences in samples rather than similarities; however, since there is no test to test for similarities, here we use it by inferring the failure to refute the null hypothesis provides evidence towards the similarities of the results. The test returns a p value of 1, meaning from the evidence presented, the hypothesis cannot be rejected. Although this does not prove the null hypothesis, it is felt the evidence is enough to validate the model.

9.7 Adding a social network

With the network implementation validated, a social network is added, allowing investigation into the effect of in-group homogeneity. In Schelling's model, agents can decipher between type but are unable to further differentiate, *i.e.*, all agents of the same type are given equal weighting. This assumption of binary differentiation is apparent in all Schelling models,

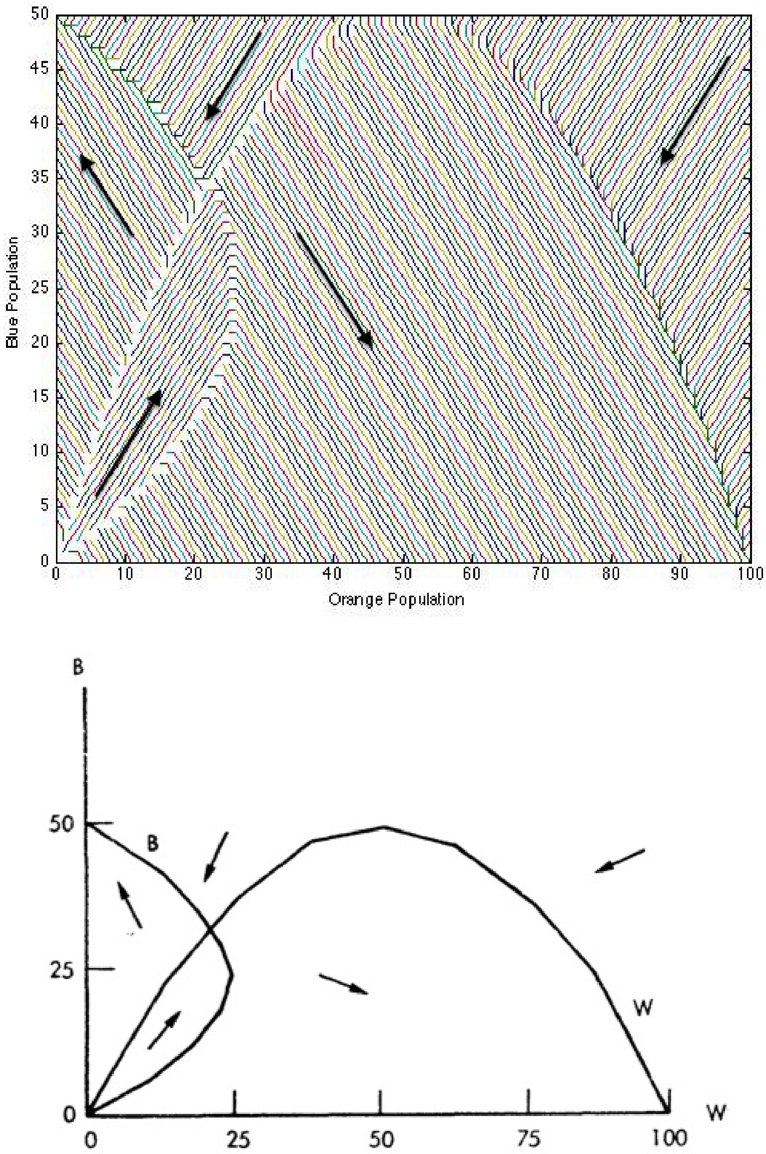


Figure 9.4: Comparing the dynamic results of the network implementation against the results from Schelling [74].

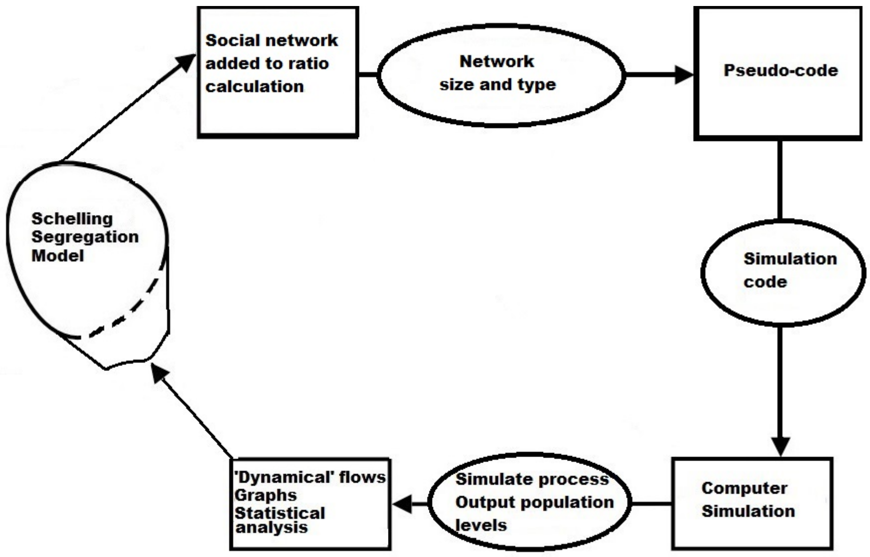


Figure 9.5: Testing binary differentiation in Schelling's Bounded Neighbourhood Model with the addition of a social network

and yet remains to be tested. To test the effect of this assumption, a social network is added, adjusting the effect of individuals connected. The addition of a social network means the creation of a new set of connections. Following Section 2.4, m defines the connections between n agents inside the neighbourhood; initially agents inside a social network (n_s) are connected by m_s links. For convenience, the social network is initialised with n nodes on a 1 dimensional lattice, with periodic boundary conditions (*i.e.*, a circle). Agents are connected to their m_s nearest neighbours, creating an initial network of n_s agents, with m_s connections. The agents are arranged along the lattice by type but not, necessarily, by tolerance. There are many possible networks that could be applied, but work here focuses on two: the Erdős-Rényi (or random) network [32] and the Watts-Strogatz (or small-world) network [96]. The random network is chosen as the simplest form of network to generate. In the random network, agents rewire each of their m_s connections, with probability $p = 1$, to randomly chosen other agents in the network. So, in this case, agents, who were mainly connected to their own type, are now connected with the distribution

$$p_k = \begin{cases} 0.67 & \text{link to } o \\ 0.33 & \text{link to } b. \end{cases} \tag{9.7}$$

The small-world network is chosen as it is an interesting variation of the random network.

For the small-world network, the initial connections are rewired according to a probability $p = 0.33$. Thus the initial bias in the social networks is maintained, but one third of the agents' connections will be randomly distributed as described previously. Simple examples of the initialisation and implementation of the two types of networks and their resulting m_s are shown in Figure 9.6. The small world network retains more of the order, with agents more likely to be connected to their own type. The random network connections follow the random distribution, with agents more likely to be connected to an agent of type O.

9.8 Hypothesis

The hypotheses tests the effects of adding a social network to the model. We offer the hypothesis that changing the movement will have an effect on the outcome of the simulation. This is tested by attempting to refute the null hypothesis (H_0), that the change has no effect, using two measures, static and dynamic (discussed in Section 4.4).

9.8.1 Static measure

The static measure of the final population inside the neighbourhood, shows whether the alteration to the model produces stable mixed populations. Following equation (4.1), M is used to measure the final population configurations. The hypothesis, that changing the movement rule will produce mixed populations, is tested by comparing M to Schelling's findings of $M = 0$. A result of $M > 0$ suggests mixed populations have arisen, counter to Schelling's model.

9.8.2 Dynamic measure

The analysis of the population flows focuses on I, which is the number of iterations taken to reach a steady state. The system has reached a steady state when no more agents wish to move. The results are analysed using the Mann-Whitney-Wilcoxon U test and the Vargha-Delaney A-statistic, described in Section 4.4.

9.9 Domain model: adding a social network

The addition of a social network necessitates a change in the calculation of R. Agents of the same type inside a social network are counted double, if and only if they are connected to the neighbourhood component. Agents of the other type, inside a social network and connected to the neighbourhood component are negated. As with Schelling's model, agents

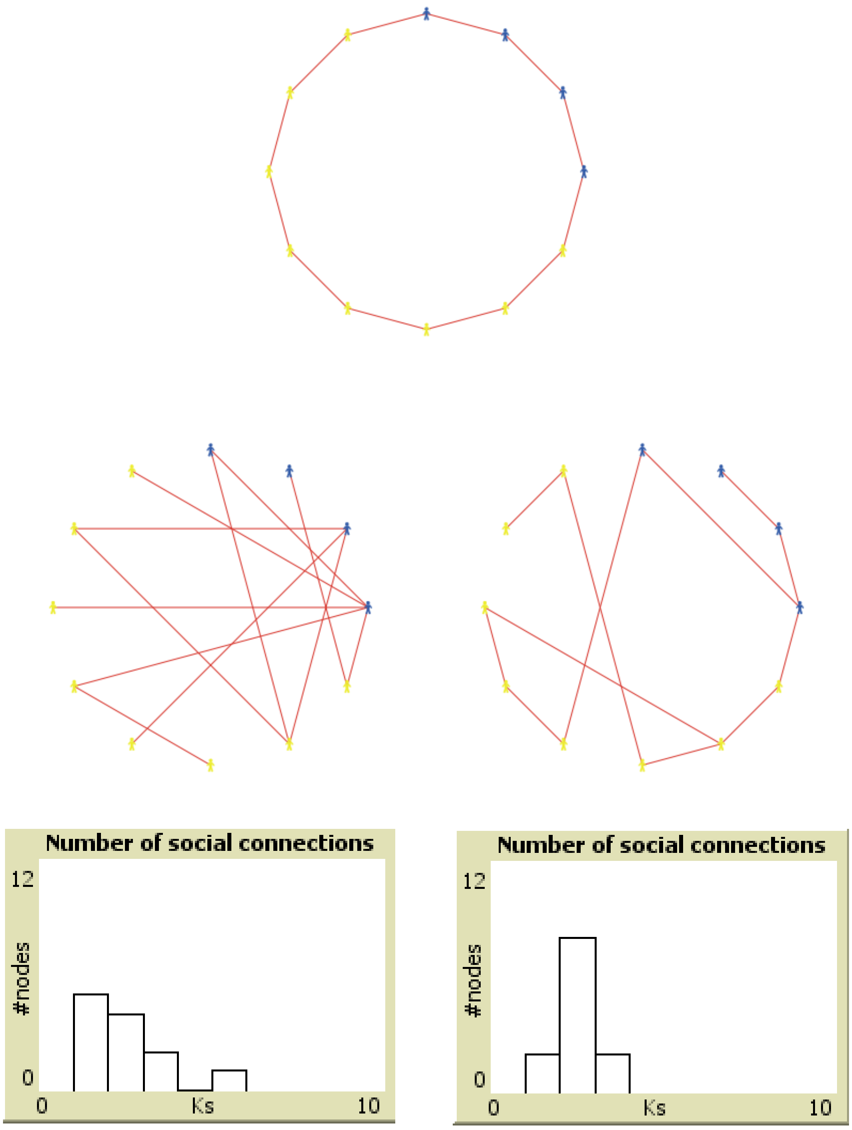


Figure 9.6: Comparison of social networks and their resulting m_s. Networks are initialised with $n = 12$ (8 orange and 4 blue) and with $m_s = 2$ (top). Connections are then rewired, with $p = 1$ for a random network (*bottom left*) and $p = 0.33$ for a small world network (*bottom right*)

Algorithm 16 Bounded Neighbourhood Network Model Initialisation

```
Create population of n agents
apply type to n agents
apply τ to n agents
for each agent n do
  Create m_s social connections with nearest neighbours
  rewire m_s with probability p
end for
fully connect n' ≤ n agents with m links
while any n ∈ n'∧ [sad] or n ∋ n'∧ [happy] do
  execute agent update (Algorithm 15)
end while
```

not connected to the neighbourhood network are ignored. The ratio calculation for the O population changes from equation (9.5) to

$$R_{BO} = \frac{n'[B] - n'[B \wedge friend]}{n'[O] + n'[O \wedge friend]}. \tag{9.8}$$

Similarly the calculation for the B population is

$$R_{OB} = \frac{n'[O] - n'[O \wedge friend]}{n'[B] + n'[B \wedge friend]}. \tag{9.9}$$

All agents are initialised with a specific social network, containing m_s links to other agents. Unlike the neighbourhood network, once rewired, the social network remains unchanged for the duration of the run. The dynamics of the model, once initialised, are unchanged (see Figure 9.7).

9.10 Platform Model

The Platform Model is described using the pseudo-code in algorithm 16, from which the simulation is constructed. Figure 9.7 gives an edited example of a simulation run of the small world network from Figure 9.6.

9.11 Simulation Platform

Since a network model has already been developed, the Simulation Platform is retained from the original network experiment. This means the addition of a social network is a simple step of creating a set of 'social' links, that will sit alongside the 'neighbourhood' links.

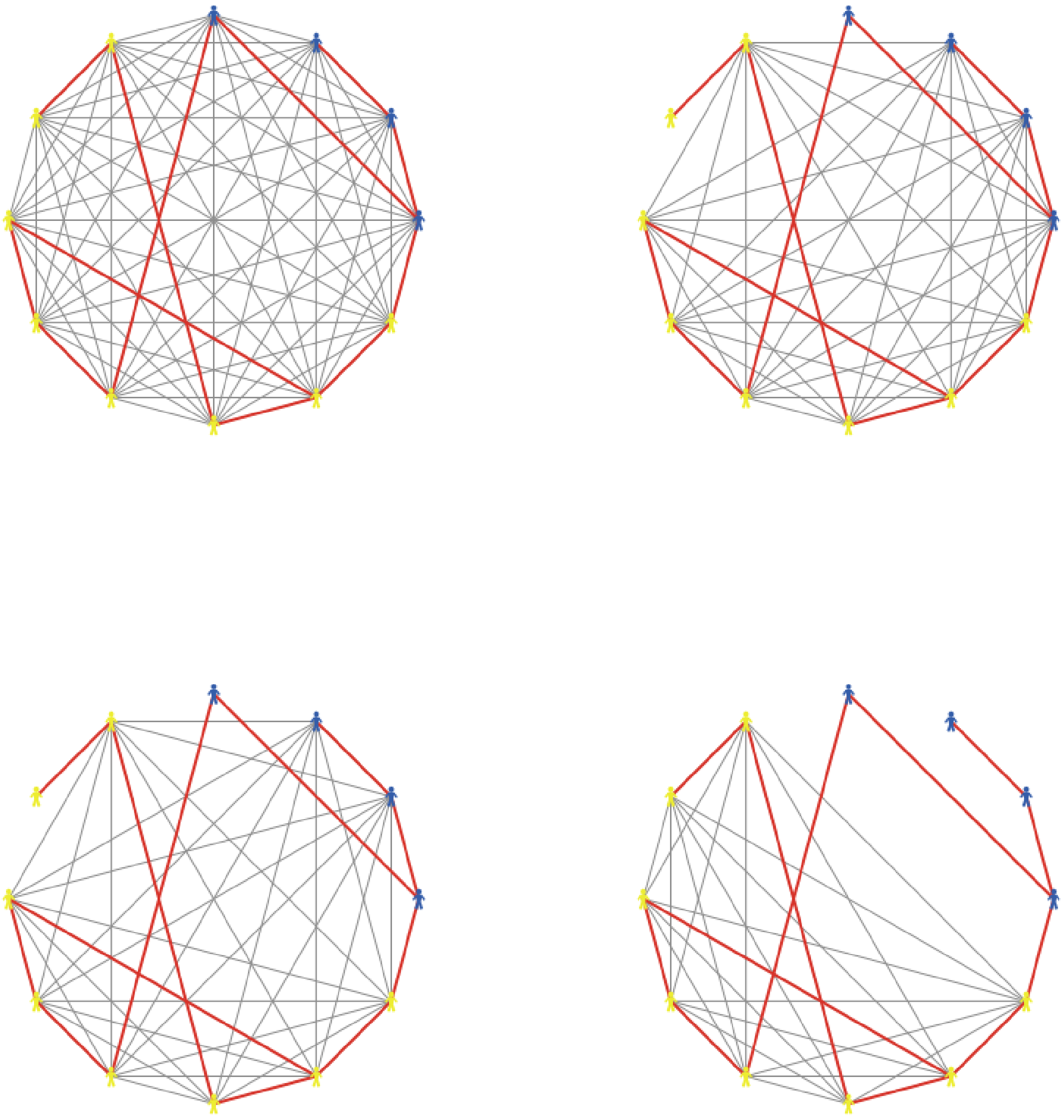


Figure 9.7: An example run of the small world social network from Figure 9.6. The social network (red links) remains unchanged over the life of the run. The changes in the neighbourhood links (grey) follow Schelling's rules. Those inside the neighbourhood (n') are connected by grey links. Those inside the social network (n_s) are connected by m_s red links. As agents connect and disconnect from the neighbourhood, the ratio changes. This leads to a final state (*bottom right*) where no agents are willing to change their connection to the network with, $n = 12, n' = 8, m = 28$ and $\rho = 7/12$. The p_k distribution of the social network is shown in the bottom right of Figure 9.6. At the end of the run, only the orange agents remain connected to the neighbourhood network; the social network has failed to stop segregation in this instance.

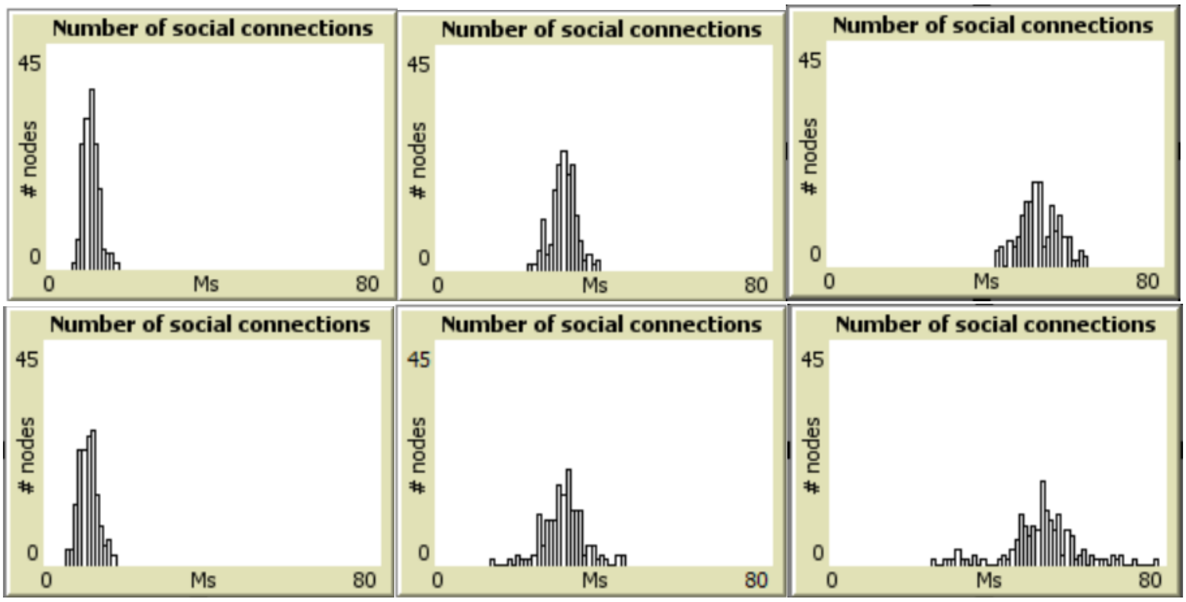


Figure 9.8: The distributions of six of the networks used to test the effect of social networks.Three small world (top) and three random (bottom). With (*from left*) initial $m_s = 10, 30$ and 50.

9.12 Results Model

This experiment tests the dynamic hypothesis that the social network makes no difference in the number I of iterations taken to reach a steady state. Again, this is tested using the U test (described in Section 4.4.3). The initial conditions of the neighbourhood network remain the same as the validation test in Chapter 6. Several values of m_s are tested, $m_s \in \{10, 20, 30, 40, 50\}$. The social network rewires according to p, with $p = 1$ for the random network and $p = 0.33$ for the small world. For each value of m_s and p, three specific networks are constructed and tested over a number of runs. Consistent with the previous experiment, all 5000 possible initial conditions are tested using these parameters. Figure 9.8 shows the m_s values for a selection of the networks used. The static analysis measures the configuration (M) of the population, to check for stable mixed populations, with the results displayed in Table 9.1. The number of iterations taken to reach a steady state is also recorded, with the resulting distribution analysed against the validated results. The results are used to test the null hypothesis, that the addition of a social network has no effect on the distribution of I. This is tested using the U test described in Section 4.4.

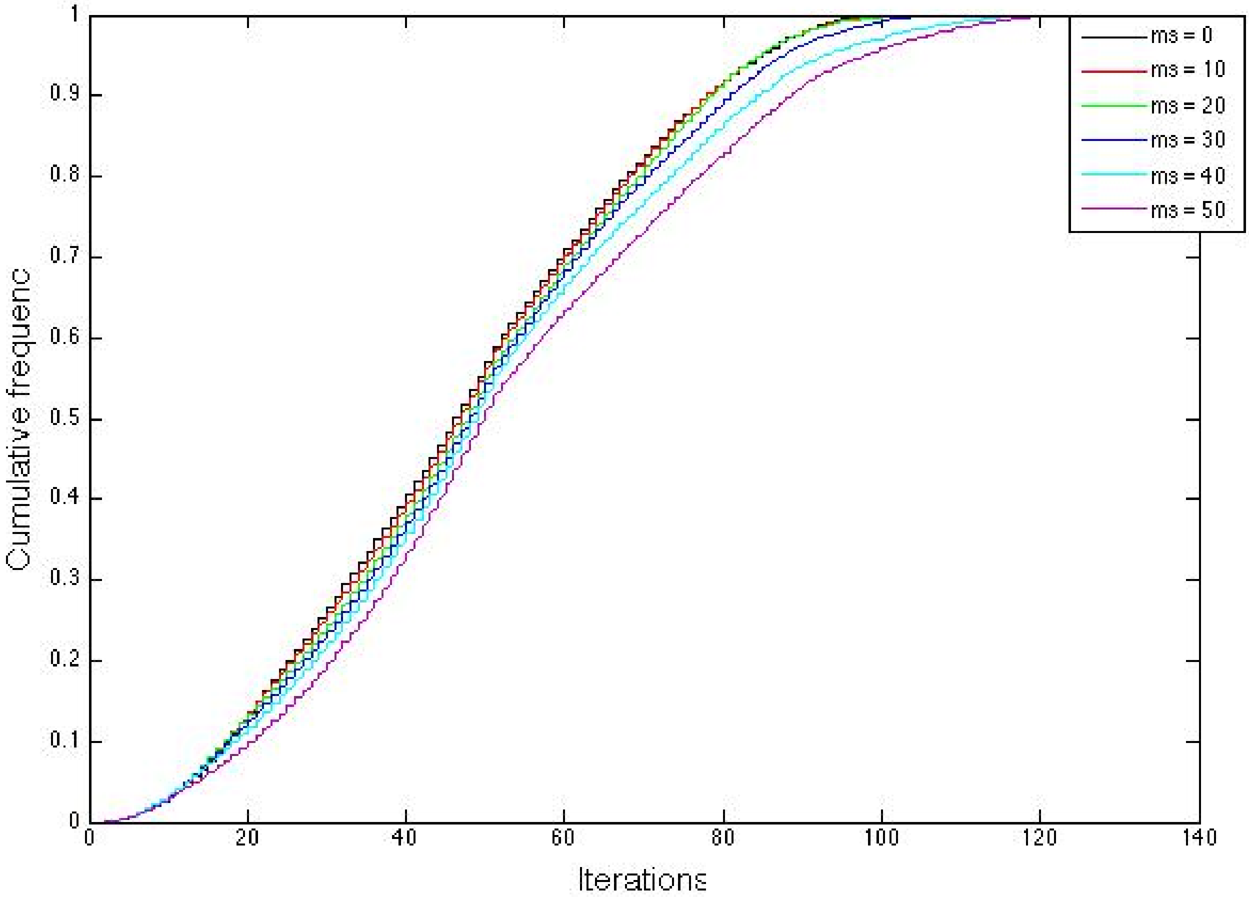


Figure 9.9: Comparing runtime distributions of different size small world social networks. The distributions of I in the small world networks appear similar. There is little noticeable difference between the validation results (in black), when compared to the results from small world networks with $m_s < 40$. The difference becomes more noticeable when $m_s \geq 40$, results which are confirmed by the U-test (see Table 9.1).

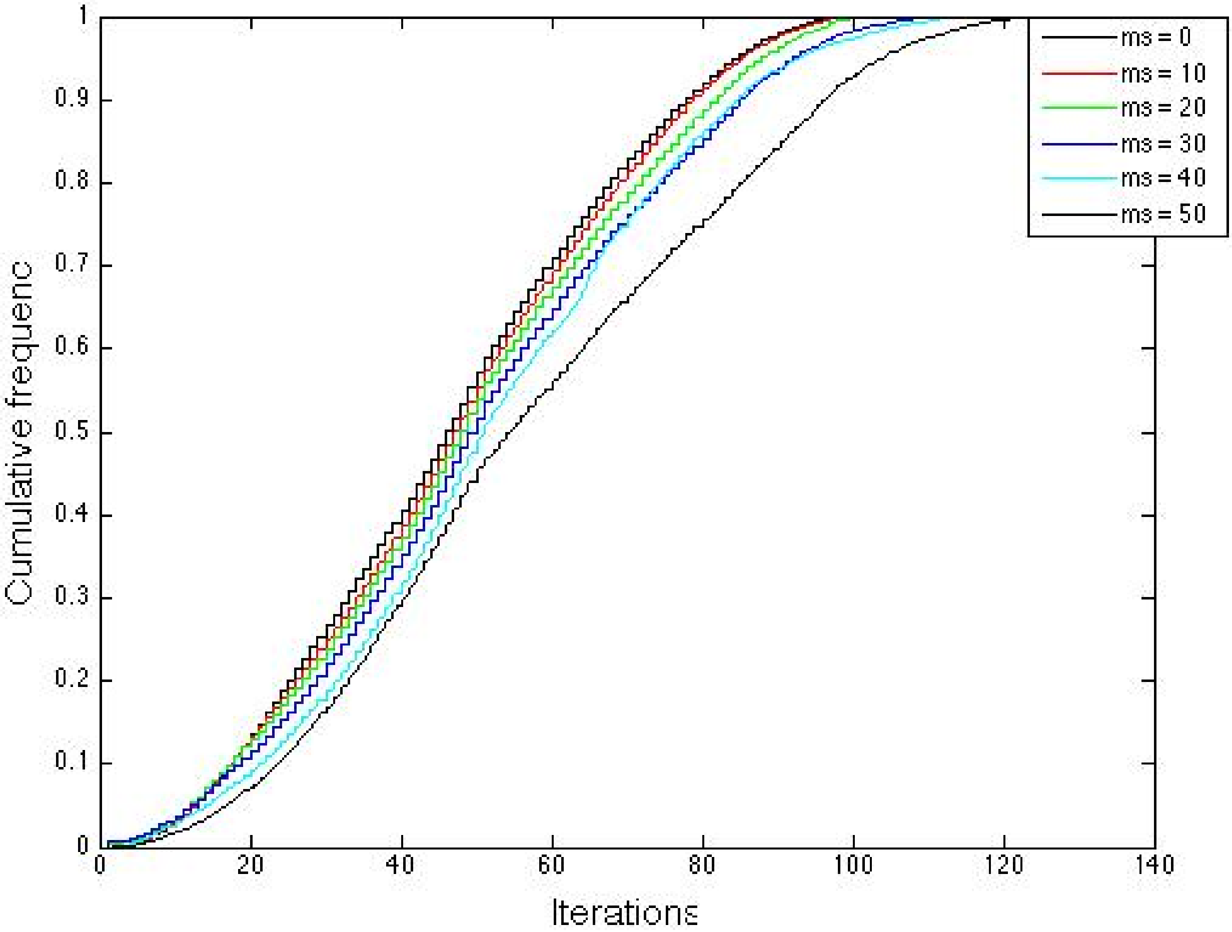


Figure 9.10: Comparing runtime distributions of different size random social networks. Here the difference between the random network and validation results are more pronounced. Indeed the results for $m_s = 50$ are noticeably different, a fact highlighted in the results of the analysis.

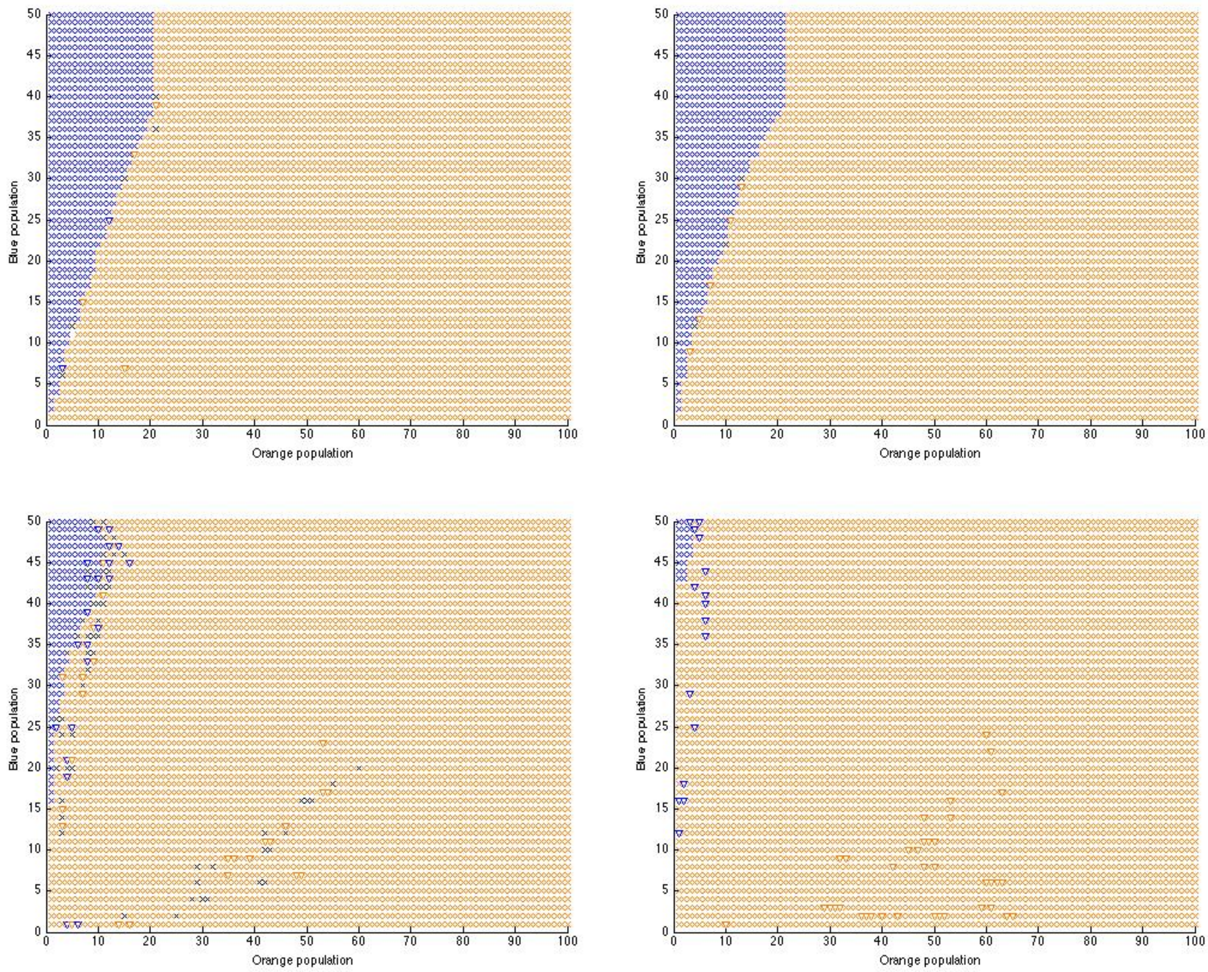


Figure 9.11: Comparing final neighbourhood network configurations of small world (left) and random (right) networks, with $m_s = 10$ (top) and $m_s = 50$ (bottom). Blue and orange results show the system has completely segregated. Crosses (x) indicate the entire population (blue or orange) has entered the neighbourhood. Circles (o) indicate the population is entirely blue or orange but there are still agents of that type outside the neighbourhood. Triangles show mixed populations; however, $R > 2$. But the black crosses (x) (bottom left) indicate final mixed populations with $R < 2$ *for both populations.*

9.12.1 Appearance of mixed populations

Following work by Freeman [38], we define m_n^* as neighbourhood links connecting agents of different type (*i.e.*, blue with orange and vice-versa). The measure M of the segregation of the population is now given as

$$M = \frac{m_n^*}{m_n}, \tag{9.10}$$

thus $M = 1$ implies an equal mix and $M = 0$ complete segregation. Summing all the M values and dividing by the number of samples in each run returns the mean M value. The results are displayed in Table 9.1. Positive values of M imply that mixed populations exist; using their initial conditions these points are mapped onto Figure 9.11. Since previous results of $M > 0$ were simply artefacts of the system reaching the maximum allowed number of iterations, it is important to show that these populations are not in the process of segregating. This is refuted by showing all the populations reaching a steady state (whereby no agents wishes to move), before the number of iterations reaches the maximum. As Figures 9.9 and 9.10 show, none of the runtimes come close to the maximum value of I. The population flows show mixed population that have arisen not as an artefact of, either, an initial condition, or the maximum I (as with Chapter 8).

The analysis of the runtime distributions highlights a difference for both network sizes and types (figures 9.9, 9.10). Statistical analysis suggests that, whilst the null hypothesis can be rejected for social networks initialised with $m_s \geq 10$; there is no effect size until $m_s \geq 20$ on the random network, and none for the small world. While this suggests that the significant results for $m_s < 20$ are caused by the sample size, we have to reject the null hypothesis. However the A statistic shows no effect size for the small world networks and only appears when $m_s \geq 20$. A representative sample of results are displayed in table 9.1. Results with an effect size are displayed in bold.

Although the social networks have to reach a relatively large size before having a noticeable effect on the simulation runtime, both types of social networks are able to maintain stable mixed populations ($M > 0$) for smaller values of m_s. This is the first time the Bounded Neighbourhood Model has produced stable mixed populations using Schelling's original parameters. These results are shown in red in Figure 9.11.

In addition, the density ρ of the mixed neighbourhood networks gives an indication of the stability of the network. A network with a higher density is less likely to be affected by the changes caused by a single agent moving. Plotting M against ρ gives a clearer picture of the neighbourhood network. As can be seen in Figure 9.12, the highest value of ρ corresponds to the lowest value of M. When $M = 0, \rho = 0.66$ or 0.33, corresponding to the neighbourhood

m_s	p(**rewire**)	Median I	U-**test**	A_{12}	M
0	-	42	1	-	0
10	1	47	$\mathbf{p < 0.05}$	0.51	0.0006
10	0.33	42	$p = 0.31$	–	0.0011
20	1	48	$\mathbf{p < 0.05}$	0.53	0.0015
20	0.33	47	$p = 0.06$	–	0.0039
30	1	50	$\mathbf{p < 0.05}$	0.55	0.0022
30	0.33	48	$\mathbf{p < 0.05}$	0.52	0.0042
40	1	50	$\mathbf{p < 0.05}$	56	0.0038
40	0.33	49	$\mathbf{p < 0.05}$	0.54	0.0047
50	1	55	$\mathbf{p < 0.05}$	**0.61**	0.0071
50	0.33	50	$\mathbf{p < 0.05}$	0.56	0.0063

Table 9.1: Results from the different networks. Significant results from the analysis of I are shown in bold. Although significant results are returned for all but two of the networks, the A-test provides evidence that these results are caused by the number of data samples. In this case, the only measurable effect size was from the random network with $m_s = 50$. However all the networks initialised were able to maintain stable mixed populations (*i.e.*, mixed populations in which all members have $\tau \geq R$), with $M > 0$ for all networks.

network either being all orange or all blue respectively. It is interesting to note that ρ increases as the size of the social network increases, however M gets smaller. This suggests that larger networks are able to sustain populations with higher levels of segregation.

9.13 Conclusions

By translating the Bounded Neighbourhood Model as a network, we have been able to add a social network, allowing investigation into the effect of binary differentiation. The results show that integrated populations are possible within the Bounded Neighbourhood Model. In addition, according to the Vargha-Delaneytest results the random social network significantly changes the time taken for the simulation to reach a steady state (although the effect size is small). As m_s increases, fewer runs terminate with the minority population inside the neighbourhood. In fact when $m_s = 50$ and $p = 1$, the minority population are unable to maintain a segregated population, even when the initialisation is heavily their favour (lower right diagram in Figure 9.11). The majority population manages to maintain itself. This is because, when the social network is the size of the minority population (*i.e.*, 50), the majority population will be able to ignore them completely ($50 - 50 = 0$). In contrast, the minority population can, at best, ignore only half of the majority population ($100 - 50 = 50$).

9.13.1 Appearance of integrated populations

A number of mixed populations remained unchanged over the runtime. Although these fixed points existed, no other populations flowed into them. This suggests they are fixed points that are difficult to reach from other initial conditions. The fact that the small world network was unable to match the effect of the random network, suggests the bias of type in the social network has an effect and is certainly an area for further exploration. The creation of mixed steady states ($M > 0$) is a novel result from the Schelling model. This provides evidence that the model can support mixed populations without explicitly altering the tolerance or population parameters. It is interesting to note that, although the small world network does not cause a significant difference in the runtime, it results in more integrated populations than the random network ($S_{sw} > S_r$). The density of the final networks has a maximum of $\rho = 0.66$, which corresponds to a network with 100 out of 150 agents connected. Thus, although the social network is able to produce mixed populations, it is unable to create mixed neighbourhood networks with a density higher than 0.6. However, the fact that the simulation produces mixed populations at all is an important result, since this is the first time Schelling's Bounded Neighbourhood Model has produced stable integrated populations.

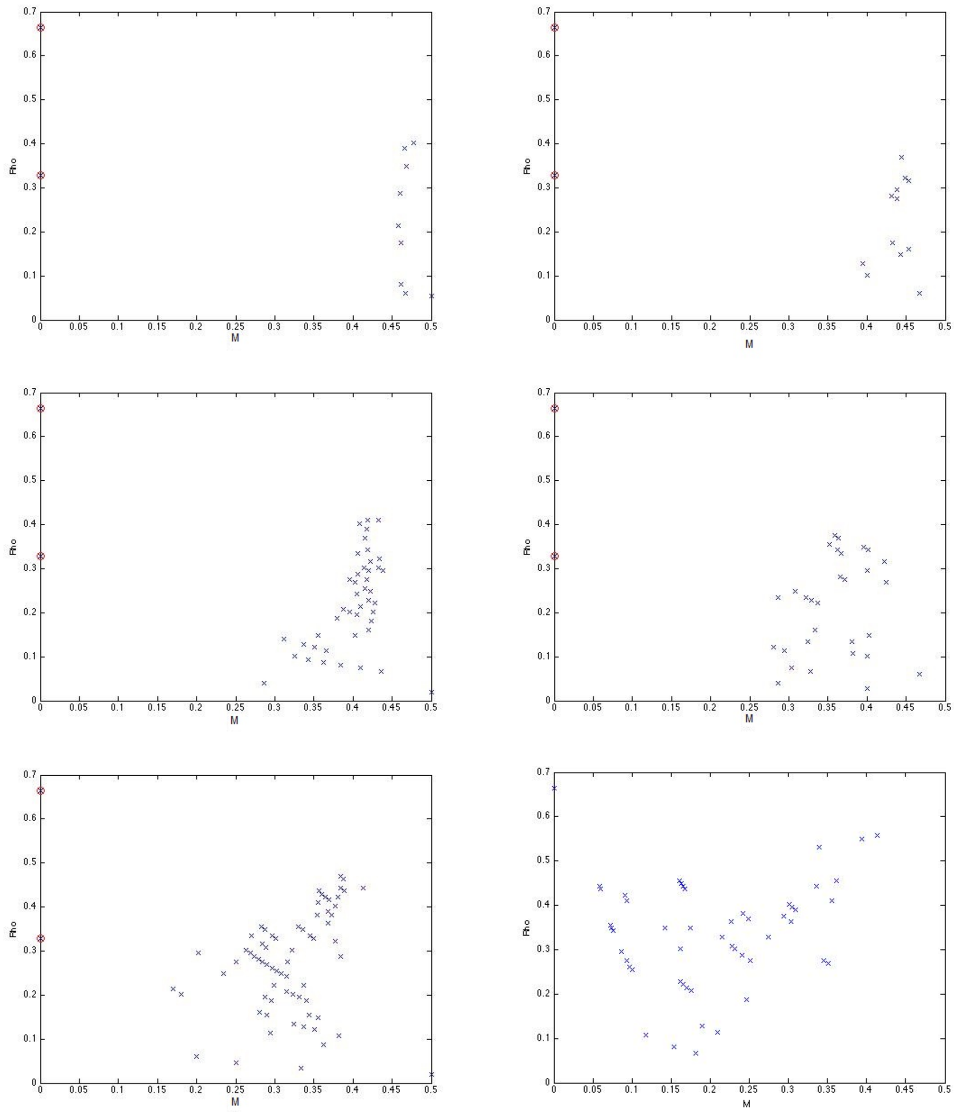


Figure 9.12: Comparing final neighbourhood network configurations (M) against density (ρ). Small world (left) and random (right) networks, with $m_s = 10$ (top), 30 (middle) and 50 (bottom). Note that as m_s increases, ρ increases and M decreases. The number of neighbourhood networks with $M > 0$ increases however, the majority of neighbourhood networks still segregate ($M = 0$). The segregated populations are highlighted with a red circle, with $\rho = 0.33$ equating to the Blue population, and $\rho = 0.66$ the Orange. It is interesting to note that as the network size increases, it is able to sustain populations with higher levels of segregation.

Chapter 10

Conclusions and Further work

Schelling's seminal segregation models helped to found the field of Computational Social Science. With well over 2000 citations, the influence of the models is matched only by Sugarscape [4]. However, the majority of work on his models has focused on the Spatial Proximity Model, where the environment is represented as a grid of cells (discussed in Chapter 3.1). This thesis has explored the less well known, but no less interesting, Bounded Neighbourhood Model. Schelling himself asked why there has been so little research into the Bounded Neighbourhood Model [76]. Claims by Hatna *et al.* that the Bounded Neighbourhood Model has been "analytically investigated" [46, p6], by Schelling [75] and Clark [23], [24] may be the explanation. However the investigation by Clark was limited to discussing the Bounded Neighbourhood Model as a variant of the Spatial Proximity Model. Indeed in [24], the model developed is an exploration of the "Schelling segregation model", with no consideration of the Bounded Neighbourhood Model. His argument, that the model has been mathematically solved, is based on Zhang's game theoretical description of the Spatial Proximity Model, in which the tolerance is the same for all individuals in a particular population, and so does not apply to the Bounded Neighbourhood Model. It is also possible that there is confusion over the models that has arisen from the terms 'Schelling segregation' and Spatial Proximity Model being used interchangeably. Fossett [37] claims his simulation "SimSeg" is a simulation of a combined Spatial Proximity Model and Bounded Neighbourhood Model. Although adjusting the spatial parameters can result in a bounded neighbourhood on the system, the agents' tolerance remains identical, as required by the Spatial Proximity Model. Although Fossett's model can vary the tolerance between groups (*i.e.*, blacks are more tolerant than whites), there is no variation within the groups (*i.e.*, all blacks have the same tolerance). However one of the defining characteristics of the Bounded Neighbourhood Model is individually variable tolerance, in which each agent within a group has a unique level of tolerance (see Section 3.2).

The analyses presented here have shown that the behaviour of the Bounded Neighbourhood Model exhibits a remarkable resistance to changes in the parameters. For many (e.g., [15], [37], [23]) this is a vindication of the strength of the model. However, applying the model to real world segregation requires a much greater understanding of the model (not to mention reality). Nevertheless, taken as a tool to explore ideas of segregation, Schelling's model has been widely used as an example of how even relatively low levels of intolerance, in a simple population, can create highly segregated populations. Attempts to apply Schelling's model to real world areas has, rightly, created much debate. Benenson and Hatna [15] and [46] use Schelling's model to explain segregation in certain Israeli cities. However, as with many segregation models that use Schelling's model, some of Schelling's assumptions are ignored. Indeed more assumptions are added in [15] (see Section 2.6). In addition there is rarely any consideration of individual adaption in the models. Thus, even though the system changes over time, the agents can only change their location. An obvious comparison is Sugarscape where, a simple inheritance rule allows adaption, such as the aggregation of wealth. However, Schelling's segregation models involve no adaption of the agents' behaviour at all.

10.1 Summary of Results

The work in this thesis has examined Schelling's Bounded Neighbourhood Model. No attempt has been made to relate it to any 'real-world' social system. Instead, following the CoSMoS process, experiments have been performed to first understand, and then probe certain key aspects of the model. This process eventually led to the development of the network model (chapter 9) which exhibited the emergence of stable mixed populations. The development of the formalisation brings out some hitherto unexplored assumptions in the Bounded Neighbourhood Model (Chapter 5) and the results from the simulation were used to validate further variations of the model. Schelling's four basic assumptions, namely movement order (Chapter 6), perfect knowledge (Chapter 7), perfect response (Chapter 8) and homogeneous populations (Chapter 9) are set out and their analysis briefly reviewed below.

For each experiment, two measures were used to test the model. One static, analysing final population mixtures, and one dynamic, analysing population movement over time. The static measure calculated the final population configurations of Schelling's model, using a parameter M that defines the mix of the population (where $M = 1$ indicates a 50/50 mix). This type of final state measure is often used when analysing Schelling segregation models (for example [15],[17] and [53]). However, in reality, systems are in constant flux, making an analysis of the dynamics of the system desirable. By measuring the number of iterations the system took to reach equilibrium, the dynamics of the model can be analysed (see Section 4.4). In addition, taking two perspectives in the analysis gives a better understanding of the model.

10.1.1 Changing the movement order: Whose move is it anyway?

In Chapter 6, Schelling's assumption that agents moved in order of their tolerance was examined. Using the formalisation and simulation developed in Chapter 5, a validated change to the movement rule was implemented. By choosing the agents in order of tolerance, the simulation replicated Schelling's flows; while choosing at random produced more chaotic flows (Figure 6.3). The effects of the change were measured and analysed using a U test (see Section 4.4) on the hypotheses:

Static measure M

The static measure analysed the robustness of segregation in the model, in response to changes in the movement order. In this experiment, changing the order of the movement had no discernible effect on the final population mix, all populations ended with complete segregation ($M = 0$).

Dynamic measure I

The dynamic hypothesis tested the number I of iterations taken for the simulation to reach a 'steady state' (as defined in Section 4.4). When agents were chosen to move at random, the number of iterations increased. The U test returned a significant result with the follow up A statistic returning a large effect size. This confirms the expectation that the time taken to reach a steady state can be increased by randomising the movement of the agents.

Results of the simulation: $M = 0$, but increased I.

Although the median value of I for the random movement was larger than the deterministic model, for both cases, every steady state was completely segregated with $M = 0$. The results from the experiment (summarised in table 10.1) show that the final populations always segregate regardless of the order of movement, although the number of iterations to reach segregation is increased.

	I	M
Deterministic	41	0
Random	49	0

Table 10.1: Summary of results from changing the order in which agents move. I represents the median number of iterations the system has taken to reach a steady state. M represents the mix of the populations in the steady state. Randomising the order of movement significantly increases I, however there is no change in the outcome of the final mix.

10.1.2 Sampling the neighbourhood: Who do you know?

In Chapter 7, the effect of the assumption that agents had perfect knowledge of the ratio R was tested. The adjustment to the model limited the number of agents S used to calculate a sample ratio R^*. By reducing S, the response of the agents is no longer perfect which, it is believed, could alter the results of the simulation. The change was implemented using the formalisation developed in Chapter 5. The modified formalisation was used to develop a Platform Model and then a simulation. The results were analysed using the parameters I and M.

Static measure M

The mix of the final populations was resistant to the changes in the model. Although mixed populations arose within the timescale, further analysis showed they were simply in a state of transition to total segregation. Increasing I_{MAX} removed the artefacts, with all populations eventually segregating.

Dynamic measure I

The number of iterations I increased as S decreased. The U test returned a significant result, providing evidence that the time taken to reach a steady state is increased when the sample size is reduced. This increase resulted in a number of results failing to reach a steady state when $S \leq 25$ (discussed below).

Robustness of the model: $M > 0$ and increased I, but final mixed populations $M > 0$ are artefacts of the length of the simulation

The results (summarised in table 10.2) demonstrate that the model reverts to Schelling's original results when $R^* \to R$ ($S = 150$). As R^* decreases, I increases, but no mixed

S	I	M
150	41	0
75	42	0
50	43	0
25	45	0
10	50	0

Table 10.2: Summary of results from sampling the neighbourhood. S is the size of the sample, I is the (median) number of iterations and M is the (mean) final population mix. As expected, reducing S increases I. However for all cases $M = 0$.

populations appear ($M = 0$).

10.1.3 Slowing agent movement: Should I stay, or should I go?

In Chapter 8, the assumption that agents leave the neighbourhood the moment they were 'sad' was tested. Using ideas from simulated annealing, agents were programmed to attempt to minimise their 'sadness' u. This parameter was based on a comparison of a second neighbourhood. The system dynamics are 'controlled' by a parameter F, used to slow the dynamics of the model. The change was implemented following the CoSMoS process, with the new formalisation developed in Chapter 8.3. The results from the simulation were analysed using the same techniques as previously, but the addition of a second neighbourhood meant that the validation run did not replicate Schelling's flows.

Static measure M

The final population was robust to changes in the perfect response of agents. Again, although mixed populations arose within the simulation timescale, all populations eventually segregated.

Dynamic measure I

As expected, the number of iterations I increased with increasing F. The U test returned a significant result providing evidence that increasing F slows the population flow. This was supported by the A-test calculating a large effect size.

F	I	M
0	149	0.005
1	220	0.005
5	588	0.005
10	1088	0.005
20	2214	0.005

Table 10.3: As the friction parameter (F) increases, the number of iterations (I) significantly increases. The appearance of mixed populations ($M \geq 0.005 \; \forall \; F > 0$) is an artefact of the model. The value M = 0.005 is an artefact caused by the symmetry of the neighbourhoods (see Section 8.6.1). Although populations with $M > 0.005$ did appear, they were simply in the process of segregating (see Chapter 8.7), and vanished when the maximum I was increased.

Results of the simulation: $M > 0$ and increased I, but final mixed populations $M > 0$ are artefacts of the length of the simulation and the ratio calculation in the model.

The results from the experiment are summarised in Table 10.3. The symmetry of the neighbourhoods means that initial conditions that have equal neighbourhood ratios, prevent the agents from moving. This means that $M \geq 0.005$ for all values of F is simply an artefact of the model and does not indicate the existence of stable mixed populations. The higher results for $F \geq 5$ are caused by the process of segregation being incomplete. This is most obvious with the median value of I for $F = 20$, which is the maximum number of iterations. Increasing the maximum value of I leads to the mixed populations eventually segregating.

10.1.4 Creating a social network: Friends and Neighbours

In Chapter 9 the assumption that agents were only discernible by type was tested. The change introduced into Schelling's Bounded Neighbourhood model exploits the idea that social networks influence behaviour, an idea of are growing interest within social modelling (for example [18] and [66]). By treating the Bounded Neighbourhood Model as a network, it is possible, indeed natural, to overlay the original network with a social network m_s and analyse the segregating of the two populations. The social network has two parameters: a size, m_s, relating to the number of nodes in the network, and a rewiring probability p. Social networks are initialised with nodes, arranged by type, connecting to their m_s nearest neighbours. Once all nodes have connected, they rewire with probability p. By choosing certain values of p, we create random and small world networks. By initialising the social network with $m_s = 0$, Schelling's original model is replicated. Other choices of assigning the connections of nodes are clearly possible and it is hoped to pursue this research in due course.

Static measure M

In previous experiments, mixed populations appeared as artefacts of the simulation finishing before the system had reached a steady state. This was reflected in large values of I. However, in this case, *stable* mixed populations appeared before the system has reached a steady state and remain mixed in the steady state. The small world networks were able to produce more integrated populations than the random networks, until $m_s = 50$. When $m_s = 50$, the random network produces more integrated populations.

Dynamic measure I

The number of iterations I increased as the network size increased. Random networks had a larger effect on I than the small world networks. However, the increase in I is the smallest across the experiments. This suggests that the networks have little effect on the flows of the populations in the model.

Robustness of the model: $M > 0$ and increased I

The results (summarised in table 10.4) indicate that random networks can sustain an increasing number of iterations without segregating and still retain mixed populations. On the other hand, small world networks (when $m_s < 50$), produced more integrated populations. Since the small world networks were more likely to contain agents of the same type (see Chapter 9), the results provide evidence that increasing the number of agents of a different type in a social network is less effective in maintaining mixed populations than increasing the number of 'same' agents.

10.2 Summary of contributions of this thesis

The work presented in this thesis has explored one of the seminal models in Computational Social Science. We have highlighted some of the possibilities computational processing brings to social science. However, importantly, we have highlighted the need for caution in these approaches. One of the biggest criticisms levelled at computational models is their lack of validity. CoSMoS provides a process that can help improve the quality of these models. By applying the process retrospectively to Schelling's Bounded Neighbourhood Model, the dynamics that drive the model have been laid bare. The results of the experiments show,

Network	m_s	I	M
-	0	42	0
Random	10	47	0.0006
Small World	10	42	0.0011
Random	20	48	0.0015
Small World	20	47	0.0039
Random	30	50	0.0022
Small World	30	48	0.0042
Random	40	51	0.0038
Small World	40	49	0.0047
Random	50	56	0.0071
Small World	50	50	0.0063

Table 10.4: Highlighted results from the different networks, m_s is the size of the network. There was an increase in I as the size of the network increased. More importantly all the networks were able to maintain stable, mixed ($M > 0$) populations.

only the implementation of a large social network can the system maintain stable mixed populations. The success of applying CoSMoS to a social system, highlights the strength and flexibility of CoSMoSand extends the CoSMoS domain of applicability.

Formalisation of the Bounded Neighbourhood Model. The thesis presents the first formalisation of Schelling's Bounded Neighbourhood model. The application of a rigorous process to the model highlighted a number of hitherto unstated assumptions.

Development of a Bounded Neighbourhood Model simulation. The simulation developed from the formalisation is the first to reproduce Schelling's original flows (see Figure 6.3). The simulation was then used as the basis for a thorough investigation of the Bounded Neighbourhood Model.

Analysis of static and dynamic behaviours of the model. Analysis of the results focused on both the final population mix (M), and, the number of iterations taken to reach steady state (I). The analysis on these two levels showed that, although changing the population flows was a straightforward process, the production of stable mixed populations required more substantial changes to the model.

Demonstration of the robustness of the model. The results of the model are 'robust' to changes in the parameters. However, a more careful examination suggests that the results are by construction, rather than by emergence. The parameters drive the model to segregation rather than allowing segregation to emerge. This hardwiring of the model is not immediately obvious and it has taken a thorough investigation to highlight the issue. This result has important implications for the use of the Bounded Neighbourhood Model in discussions of the determinants of segregation in social studies.

Demonstration of effect of social networks on segregation in the model. The demonstration that the addition of a social network produces stable, integrated, populations is more evidence of why not to apply the model blindly to real world situations. However Schelling's Bounded Neighbourhood Model still has utility and is an excellent *segregation* model.

10.3 Further work

Clearly Schelling's models have a plethora of possible future research ideas, some of which were examined in Chapter 3.6. The work presented in this thesis is a first step in a systematic exploration of Schelling's Bounded Neighbourhood model and related models. Further work could develop some of the questions raised in this study. For example:

Heterogeneous populations. Further work could use the model developed in Chapter 9 and explore the effects of different types within populations.

Space. The network model is a first step in the introduction of space into the Bounded Neighbourhood Model. The treatment of space is an important aspect of social science (especially urban studies). However, Schelling's models have a complete disregard for anything other than uniform space. It would be interesting to examine if changing the perceptions of space in the model would effect the outcomes.

Timescales. Further exploration of the movement rule could explore the effect of agents moving at different timescales, either individually or by type.

Time decay for social bonds. Linked to timescales, would a social bond that decayed over time effect the results of the network model?

Ratio calculations Although Chapter 7 explored ideas of agents calculating different ratios from the same population, further work could explore ideas of agents retaining their calculated ratio for a period of time, rather than recalculating at each iteration?

Population adaption. The character of the members in the populations undergoes no change in Schelling's models and the Bounded Neighbourhood Model in particular.

This is a serious limitation and leads one to ask what effects an evolutionary model could have on a Schelling segregation model?

These are just a few of the many questions the formalisation and testing of Schelling's Bounded Neighbourhood Model has promted. The thorough examination presented here has given an insight into the key drivers of the model. The further work described is merely the beginning of what is sure to be a rich field of research.

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